Statement:

Let X be a random variable whose distribution function is indexed by the parameter θ. Let $(x_1, x_2, \cdots x_n)$ be a sample point in the sample space S. $L(\theta)$ is the likelihood function of the sample. We want to test

$$H_0 : \theta = \theta_0$$

against $H_1 : \theta = \theta_1$

where θ_0 and θ_1 are two specific values of θ. Let ω_0 be the subset of S such that.

$$\frac{L_1}{L_0} \geq K \quad \text{inside } \omega_0$$

$$\text{and } \frac{L_1}{L_0} < K \quad . \text{ outside } \omega_0$$

Where $K > 0$ and L_0, L_1 are the likelihood functions of the sample observations under H_0 and H_1 respectively. Then ω_0 is the most powerful critical region of the above test hypothesis.

Proof:

We are given

$$P(X \in \omega_0 / H_0) = \int_{\omega_0} L_0 \, dX = \alpha.$$

If ω is any other region satisfying $\int_{\omega} L_0 dX = \alpha$

Clearly $\omega_0 = (\omega_0 \cap \omega) \cup (\omega_0 \cap \bar{\omega})$.

and $\omega = (\omega \cap \omega_0) \cup (\omega \cap \bar{\omega}_0)$

$$\text{Then } \int_{\omega} L_1 \, dX = \int_{\omega \cap \omega_0} L_1 dX + \int_{\omega \cap \bar{\omega}_0} L_1 dX \quad \text{—— (1)}$$

and $\int_{\omega_0} L_1 dX = \int_{\omega_0 \cap \omega} L_1 dX + \int_{\omega_0 \cap \bar{\omega}} L_1 dX$ ——— ②

Subtracting ② from ① we get

$$\int_{\omega} L_1 dX - \int_{\omega_0} L_1 dX = \int_{\omega \cap \bar{\omega}_0} L_1 dX - \int_{\omega_0 \cap \bar{\omega}} L_1 dX \quad \text{——— ③}$$

Now by defination $L_1 \geq K L_0$ for the sample point,

and $L_1 < K L_0$ for the sample point outside ω_0.

$$\int_{\omega \cap \bar{\omega}_0} L_1 dX < K \int_{\omega \cap \bar{\omega}_0} L_0 dX$$

and $\int_{\omega_0 \cap \bar{\omega}} L_1 dX \geq K \int_{\omega_0 \cap \bar{\omega}} L_0 dX$.

Then we have.

$$\int_{\omega \cap \bar{\omega}_0} L_1 dx - \int_{\omega_0 \cap \bar{\omega}} L_1 dX \leq K \left[\int_{\omega \cap \bar{\omega}_0} L_0 dX - \int_{\omega_0 \cap \bar{\omega}} L_0 dX \right]$$

$$= K \left[\int_{\omega \cap \bar{\omega}_0} L_0 dX + \int_{\omega \cap \omega_0} L_0 dX - \int_{\omega \cap \omega_0} L_0 dX - \int_{\omega_0 \cap \bar{\omega}} L_0 dX \right]$$

$$= K \left[\int_{\omega} L_0 dX - \int_{\omega_0} L_0 dX \right]$$

$$= K [\alpha - \alpha] = 0$$

$$\Rightarrow \int_{\omega \cap \bar{\omega}_0} L_1 dX \leq \int_{\omega_0 \cap \bar{\omega}} L_1 dX \quad \text{——— ④}$$

Now from ③ and ④ we get $\int_{\omega_0} L_1 dX \geq \int_{\omega} L_1 dX$

Hence the Lemma.