Pivotal Method to Find Confidence Interval

Let,

$x_1, x_2, \ldots, x_n$ be a random sample of n observations selected from a population having p.d.f $f(x, \theta)$. Let, $Q = q(x_1, x_2, \ldots, x_n; \theta)$ be the function of sample observations and parameter. Now, if the distribution of Q is free of θ, Q is called pivotal quantity.

e.g., if the sample observations are selected from $N(\mu, \sigma^2)$, then $Q = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$, if σ^2 is known.

Also, $Q = \frac{\bar{x} - \mu}{s/\sqrt{n}} \sim N(0,1)$; if n is Large, where s^2 is an unbiased estimator of σ^2. Here, the distribution of Q is free of parameter μ. Therefore both the Q is pivotal quantity. Even if the sample size is small, the distribution of $Q = \frac{\bar{x} - \mu}{s/\sqrt{n}}$ is free of parameter μ.

Since, Q is the pivotal quantity, we need to find two values of Q, say q_1 and q_2, so that for $0 < 1-\alpha < 1$, $P[q_1 < Q < q_2] = 1-\alpha$. Here, for all possible samples $q_1 < Q < q_2$, iff ~~and~~
$T_1(x_1, x_2, \ldots, x_n) < \psi(\theta) < T_2(x_1, x_2, \ldots, x_n)$

then $[T_1 - T_2]$ is called $100(1-\alpha)\%$ confidence interval of $\psi(\theta)$. It is ~~notice~~ noted that, the inequality $q_1 < Q < q_2$ can be inverted to the inequality $T_1(x_1, x_2, \ldots, x_n) < \psi(\theta) < T_2(x_1, x_2, \ldots, x_n)$.

e.g. for sample ~~mean~~ ~~from~~ from $N(\mu, \sigma^2)$,
$q_1 < \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} < q_2$ can be ~~int~~ inverted into

$\bar{x} - q_2 \cdot \sigma/\sqrt{n} < \mu < \bar{x} + q_1 \sigma/\sqrt{n}$. Therefore
$\left[\bar{x} - q_2 \sigma/\sqrt{n} < \mu < \bar{x} + q_1 \sigma/\sqrt{n}\right]$ is the $100(1-\alpha)\%$ confidence interval for μ. (*)

Confidence interval for population mean (μ) of $N(\mu, \sigma^2)$.

Let, $x_1, x_2, \ldots, x_n$ be a random sample from $N(\mu, \sigma^2)$, where μ is unknown but σ^2 is known. We are required to find $100(1-\alpha)\%$ confidence interval for the parameter μ.

(*) Normally q_1 and q_2 depends on α_1 and α_2, such that $P(Q < q_1) = \alpha_1$ and $P(Q > q_2) = \alpha_2$
$\Rightarrow P(q_1 < Q < q_2) = 1 - \alpha$ where $\alpha = \alpha_1 + \alpha_2$
$\forall \theta \in \Theta$ and $\alpha_1 = \alpha_2$

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Here $\theta = (\mu, \sigma^2)$

$$\Theta = \{(\mu, \sigma^2); -\infty < \mu < \infty, 0 < \sigma^2 < \infty\}$$

We may take the function $\psi(T, \theta) = \frac{\bar{x} - \mu}{\sigma/\sqrt{n}}$ where $T = \bar{x}$ (sample mean). Since, σ is known, $\psi(T, \theta) \sim N(0, 1)$ distribution and is independent of true value of θ.

We also have $P_\theta\left[-z_{\alpha/2} \leq \frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \leq +z_{\alpha/2}\right] = 1 - \alpha$

$\forall \theta \in \Theta$

$$\Rightarrow P_\theta\left[-z_{\alpha/2} \cdot \sigma/\sqrt{n} \leq \bar{x} - \mu \leq +z_{\alpha/2} \cdot \sigma/\sqrt{n}\right] = 1 - \alpha$$

$$\Rightarrow P_\theta\left[-(\bar{x} + z_{\alpha/2} \cdot \sigma/\sqrt{n}) \leq -\mu \leq -(\bar{x} - z_{\alpha/2}\, \sigma/\sqrt{n})\right] = 1 - \alpha$$

$$\Rightarrow P_\theta\left[\bar{x} - z_{\alpha/2} \cdot \sigma/\sqrt{n} \leq \mu \leq \bar{x} + z_{\alpha/2} \cdot \sigma/\sqrt{n}\right] = 1 - \alpha$$

$\forall \theta \in \Theta$

This implies that for every set of sample observations $x_1, x_2, \ldots, x_n$ we may have a confidence interval between $\bar{x} \pm z_{\alpha/2} \cdot \sigma/\sqrt{n}$ with confidence coefficient $1-\alpha$.

Let,

$x_1, x_2, \ldots, x_n$ be a random sample from $N(\mu, \sigma^2)$ where μ is unknown and also σ^2 is unknown we are required to find $100(1-\alpha)\%$ confidence interval for the parameter μ.

Here, $\theta = (\mu, \sigma^2)$ and $\Theta = \{(\mu, \sigma^2); -\infty < \mu < +\infty, 0 < \sigma^2 < \infty\}$

we define $\psi(T, \theta) = \frac{\bar{x} - \mu}{S/\sqrt{n}}$, where $S^2 = \frac{1}{n-1}\Sigma(x_i - \bar{x})^2$ is the unbiased estimator of population variance σ^2. We know that the statistic $\psi(T, \theta) \sim t_{n-1}$ (student's) and is independent of the true value of θ. We also have $P_\theta\left[-t_{\alpha/2, n-1} \le \frac{\bar{x} - \mu}{S/\sqrt{n}} \le +t_{\alpha/2, n-1}\right] = 1 - \alpha$

$$\Rightarrow P_\theta\left[-t_{\alpha/2, n-1} \cdot S/\sqrt{n} \le \bar{x} - \mu \le +t_{\alpha/2, n-1} \cdot S/\sqrt{n}\right] = 1 - \alpha$$

$$\Rightarrow P_\theta\left[-(\bar{x} + t_{\alpha/2, n-1} \cdot S/\sqrt{n}) \le -\mu \le -(\bar{x} - t_{\alpha/2, n-1} \cdot S/\sqrt{n})\right] = 1 - \alpha$$

$$\Rightarrow P_\theta\left[\bar{x} - t_{\alpha/2, n-1} \cdot S/\sqrt{n} \le \mu \le \bar{x} + t_{\alpha/2, n-1} \cdot S/\sqrt{n}\right] = 1 - \alpha$$

$\forall \theta \in \Theta$

This implies that for every set of sample observations $x_1, x_2, \ldots, x_n$ we may obtain confidence limits $\bar{x} \pm t_{\alpha/2, n-1} \cdot S/\sqrt{n}$ with confidence coefficient $1 - \alpha$ for the parameter μ. Where $\bar{x} = \frac{1}{n}\Sigma x_i$

Let, $x_1, x_2, \ldots, x_n$ be a random sample from $N(\mu, \sigma^2)$ where μ is known but σ^2 is unknown. We are required to find $100(1-\alpha)\%$ confidence interval for the parameter σ^2.

Here, $\theta = (\mu, \sigma^2)$ and

$$\Theta = \{(\mu, \sigma^2); -\infty < \mu < \infty,\ 0 < \sigma^2 < \infty\}$$

we may take $\psi(T, \theta) = \frac{\sum (x_i - \mu)^2}{\sigma^2}$

We know that the statistic $\psi(T, \theta) \sim \chi^2_n$ and is independent of ~~true~~ the true value of θ.

Also,

$$P_\theta\left\{\chi^2_{1-\alpha/2, n} \le \frac{\sum (x_i - \mu)^2}{\sigma^2} \le \chi^2_{\alpha/2; n}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{\frac{\chi^2_{1-\alpha/2, n}}{\sum (x_i - \mu)^2} \le \frac{1}{\sigma^2} \le \frac{\chi^2_{n, \alpha/2}}{\sum (x_i - \mu)^2}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{\frac{\sum (x_i - \mu)^2}{\chi^2_{n, \alpha/2}} \le \sigma^2 \le \frac{\sum (x_i - \mu)^2}{\chi^2_{n, 1-\alpha/2}}\right\} = 1-\alpha, \forall \theta \in \Theta$$

The above statement implies that for every set of sample values $x_1, x_2, \ldots, x_n$ we may obtain the confidence interval

$$\left[\frac{\Sigma(x_i-\mu)^2}{\chi^2_{n,\alpha/2}}, \frac{\Sigma(x_i-\mu)^2}{\chi^2_{n,1-\alpha/2}}\right]$$

with confidence coefficient $1-\alpha$.

Let $x_1, x_2, x_3, \ldots, x_n$ be a random sample from $N(\mu, \sigma^2)$ where μ is unknown and σ^2 is unknown. We are required to find $100(1-\alpha)\%$ confidence interval for the parameter σ^2.

Here, $\theta = (\mu, \sigma^2)$

and $\Theta = \{(\mu, \sigma^2); -\infty < \mu < \infty, 0 < \sigma^2 < \infty\}$.

we define $\psi(T, \theta) = \frac{\Sigma(x_i-\bar{x})^2}{\sigma^2}$. This ~~meas~~ measurable function $\psi(T, \theta)$ (statistic), is a random variable following χ^2 distribution with $n-1$ d.f.

Also, we have

$$P_\theta\left\{\chi^2_{n-1,1-\alpha/2} \le \frac{\Sigma(x_i-\bar{x})^2}{\sigma^2} \le \chi^2_{n-1,\alpha/2}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{\frac{\chi^2_{n-1,1-\alpha/2}}{\Sigma(x_i-\bar{x})^2} \le \frac{1}{\sigma^2} \le \frac{\chi^2_{n-1,\alpha/2}}{\Sigma(x_i-\bar{x})^2}\right\} = 1-\alpha$$

$$\Rightarrow P_\theta\left\{\frac{\Sigma(x_i-\bar{x})^2}{\chi^2_{n-1,1-\alpha/2}} \le \sigma^2 \le \frac{\Sigma(x_i-\bar{x})^2}{\chi^2_{n-1,\alpha/2}}\right\} = 1-\alpha, \forall \theta \in \Theta$$

The above statement implies that for every set of sample values $x_1, x_2, \ldots, x_n$ we may obtain the confidence interval

$$\left[\frac{\sum(x_i-\bar{x})^2}{\chi^2_{n-1,\,\alpha/2}}, \frac{\sum(x_i-\bar{x})^2}{\chi^2_{n-1,\,1-\alpha/2}}\right]$$

with confidence coefficient $1-\alpha$.