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The motion of a pure oscillation in time variable t is expressed as

$$A \sin\left(\alpha + \frac{2\pi t}{\lambda}\right).$$

Economic variables as in business cycle often show movements which are approximately periodic with no fixed ~~base~~ period. The time series can then be expressed as the sum of a number of strictly periodic term plus an error component. Thus, such a series can be expressed as

$$y_t = \sum A_i \sin\left(\alpha_i + \frac{2\pi}{\lambda_i} t\right) + u_t,$$

where $u_t$ is an error component.

A way of analyzing a time series from which trend and seasonal variations have been eliminated is, therefore to assume that it is made up of sine and cosine waves of different frequencies such as

$$y_t = a_0 + \sum a_i \cos\left(\frac{2\pi}{\lambda_i} t\right) + \sum b_i \sin\left(\frac{2\pi}{\lambda_i} t\right) + u_t.$$

The object is to estimate the periodocities $\lambda_i$ (that is, to find out what are the best values of $\lambda$s to select) and to evaluate the constants $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ This is known as periodogram analysis, first devised by Schuster (1898).

Take a number of trial periods $\mu_1, \mu_2, \ldots$ and for each trial period, compute the value of an indicator function. The indicator is so chosen that when any of the trial periods approaches any of the true periods $\lambda_1, \lambda_2, \ldots$, the indicator function attains a local maxima. The graph of this indicator function against the trial period is known as periodogram.

Consider a simple case where $y_t$ is given by

$$y_t = a \sin \frac{2\pi t}{\lambda} + u_t .$$

$u_t$ does not contain any cyclical component and it is assumed that $Corr\left(u_t, \sin\left(\frac{2\pi t}{\lambda}\right)\right) = 0$. Take $N$ observations and a trial period $\mu$ so that $N = k\mu$ ($k$ is an integer) and arrange the observations into $k \times \mu$ array as

$$\begin{array}{cccc} y_1 & y_2 & \cdots & y_\mu \\ y_{\mu+1} & y_{\mu+2} & \cdots & y_{2\mu} \\ \vdots & \vdots & & \vdots \\ y_{(k-1)\mu+1} & y_{(k-1)\mu+2} & \cdots & y_{k\mu} \end{array}$$

which is sometimes known as Buys-Ballot table. If $N$ is not a multiple of $\mu$, writing down the rows is to be continued until there are fewer than $\mu$ terms left.

Suppose, a curve is fitted with equation

$$Z_t = A_0 + A\cos\frac{2\pi}{\mu}t + B\sin\frac{2\pi}{\mu}t$$

to the observed data. The constants $A_0, A, B$ are obtained by minimizing

$$E = \sum (y_t - Z_t)^2 = \sum \left(y_t - A_0 - A\cos\frac{2\pi}{\mu}t - B\sin\frac{2\pi}{\mu}t\right)^2$$

w.r.t $A_0, A, B$.

The normal equations are

$$\frac{\partial E}{\partial A_0} = 0 \quad \sum_t y_t = n A_0 + \sum A\cos\frac{2\pi}{\mu}t + \sum B\sin\frac{2\pi}{\mu}t$$

$$\frac{\partial E}{\partial A} = 0 \quad \sum y_t \cos\frac{2\pi}{\mu}t = \sum A_0 \cos\frac{2\pi}{\mu}t + \sum A\cos^2\frac{2\pi}{\mu}t + \sum B\sin\frac{2\pi}{\mu}t\cos\frac{2\pi}{\mu}t$$

$$\frac{\partial E}{\partial B} = 0 \quad \sum y_t \sin\frac{2\pi}{\mu}t = \sum A_0 \sin\frac{2\pi}{\mu}t + \sum A\cos\frac{2\pi}{\mu}t\sin\frac{2\pi}{\mu}t + \sum B\sin^2\frac{2\pi}{\mu}t$$

Therefore, using

$$\sum_{t=1}^{n} \sin\frac{2\pi t}{\mu} = \sum_{t=1}^{n} \cos\frac{2\pi t}{\mu} = 0$$

$$\sum_{t=1}^{n} \cos^2\frac{2\pi t}{\mu} = \sum_{t=1}^{n} \sin^2\frac{2\pi t}{\mu} = \frac{n}{2},$$

$$A_0 = \Sigma y_t / n = \bar{y}_t ,$$

$$A = \frac{2}{n} \Sigma y_t \cos \frac{2\pi t}{u} ,$$

$$B = \frac{2}{n} \Sigma y_t \sin \frac{2\pi t}{u} .$$

Let us consider,

$$A = \frac{2}{n}\sum_{t=1}^{n} Y_t \cos\left(\frac{2\pi}{\mu}\right)t \quad ; \quad B = \frac{2}{n}\sum_{t=1}^{n} Y_t \sin\left(\frac{2\pi}{\mu}\right)t$$

$$\therefore \quad S^2(\mu) = A^2 + B^2 \qquad \text{[Where, } \mu \text{ is arbitrary]}$$

Now,

$$A = \frac{2}{n}\sum_{t=1}^{n} Y_t \cos\left(\frac{2\pi}{\mu}\right)t$$

$$\Rightarrow A = \frac{2}{n}\sum_{t=1}^{n}\left\{\left(a\sin\left(\frac{2\pi}{\lambda}\right)t + \varepsilon_t\right)\cos\left(\frac{2\pi}{\mu}\right)t\right\}$$

[from eqn (i)]

$$\Rightarrow A = \frac{2}{n}\sum_{t=1}^{n} a\sin\left(\frac{2\pi}{\lambda}t\right)\cos\left(\frac{2\pi}{\mu}t\right) \qquad [\because \varepsilon_t \text{ is un-correlated}]$$

$$\Rightarrow A = \frac{a}{n}\sum_{t=1}^{n} 2\sin\alpha t\cos\beta t \qquad \left[\text{Where, } \alpha = \frac{2\pi}{\lambda},\ \beta = \frac{2\pi}{\mu}\right]$$

$$\Rightarrow A = \frac{a}{n}\sum_{t=1}^{n}\left\{\sin(\alpha+\beta)t + \sin(\alpha-\beta)t\right\} \qquad \left[2\sin a\cos b = \sin(a+b) + \sin(a-b)\right]$$

——— (ii)

<u>Let,</u>

$$S = \sum_{t=1}^{n} \sin(\alpha+\beta)t$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = \sum_{t=1}^{n}\left(\sin(\alpha+\beta)t\right).\sin\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\sum_{t=1}^{n} 2\sin\{(\alpha+\beta).t\}\sin\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\sum_{t=1}^{n}\left[\cos\left\{(\alpha+\beta)t-\left(\frac{\alpha+\beta}{2}\right)\right\} - \cos\left\{(\alpha+\beta)t+\left(\frac{\alpha+\beta}{2}\right)\right\}\right]$$

$$\left[2\sin a\sin b = \cos(a-b) - \cos(a+b)\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\left[\left\{\cos\left(\frac{\alpha+\beta}{2}\right) - \cos\frac{3(\alpha+\beta)}{2}\right\} + \left\{\cos\frac{3(\alpha+\beta)}{2} - \cos\frac{5(\alpha+\beta)}{2}\right\} + \dots + \left\{\cos\frac{(2n-1)(\alpha+\beta)}{2} - \cos\frac{(2n+1)(\alpha+\beta)}{2}\right\}\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2}\right) - \cos\frac{(2n+1)(\alpha+\beta)}{2}\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}.\left[-2.\sin\frac{\cancel{2}(n+1)(\alpha+\beta)}{\cancel{2}.2}\ \sin\frac{-\cancel{2}n(\alpha+\beta)}{\cancel{2}.2}\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = \left[\sin\frac{(n+1)(\alpha+\beta)}{2}.\sin\frac{n(\alpha+\beta)}{2}\right]$$

$$\Rightarrow S = \frac{\sin\frac{(n+1)(\alpha+\beta)}{2}.\sin\frac{n(\alpha+\beta)}{2}}{\sin\left(\frac{\alpha+\beta}{2}\right)} \quad \text{——— (iii)}$$

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Again,

$$S = \sum_{t=1}^{n} \sin(\alpha-\beta)t$$

$$\Rightarrow S.\sin\left(\frac{\alpha-\beta}{2}\right) = \sum_{t=1}^{n}\left(\sin(\alpha-\beta)t\right).\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow S.\sin\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}\sum_{t=1}^{n} 2\sin\{(\alpha-\beta).t\}.\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow S.\sin\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}\sum_{t=1}^{n}\left[\cos\left\{(\alpha-\beta)t.-\left(\frac{\alpha-\beta}{2}\right)\right\} - \cos\left\{(\alpha-\beta).t+\left(\frac{\alpha-\beta}{2}\right)\right\}\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}\left[\left\{\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\frac{3(\alpha-\beta)}{2}\right\} + \left\{\cos\frac{3(\alpha-\beta)}{2} - \cos\frac{5(\alpha-\beta)}{2}\right\} + \dots + \left\{\cos\frac{(2n-1)(\alpha-\beta)}{2} - \cos\frac{(2n+1)(\alpha-\beta)}{2}\right\}\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}\left[\cos\left(\frac{\alpha-\beta}{2}\right) - \cos\frac{(2n+1)(\alpha-\beta)}{2}\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha-\beta}{2}\right) = \frac{1}{2}\left[-2\sin\frac{\cancel{2}(n+1)(\alpha-\beta)}{\cancel{2}.2}.\sin\frac{-\cancel{2}.n(\alpha-\beta)}{\cancel{2}.2}\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha-\beta}{2}\right) = \sin\frac{(n+1)(\alpha-\beta)}{2}\sin\frac{n(\alpha-\beta)}{2}$$

$$\Rightarrow S = \frac{\sin\frac{(n+1)(\alpha-\beta)}{2}.\sin\frac{n(\alpha-\beta)}{2}}{\sin\frac{(\alpha-\beta)}{2}} \quad \text{——— (iv)}$$

$\therefore$ From eqn (ii), (iii) and (iv) we get $\rightarrow$

$$A = \frac{a}{n}\left[\frac{\sin\frac{(n+1)(\alpha+\beta)}{2}.\sin\frac{n(\alpha+\beta)}{2}}{\sin\frac{(\alpha+\beta)}{2}} + \frac{\sin\frac{(n+1)(\alpha-\beta)}{2}.\sin\frac{n.(\alpha-\beta)}{2}}{\sin\left(\frac{\alpha-\beta}{2}\right)}\right]$$

If $\alpha \neq \beta$, then $A \to 0$ for large $n$, Since then the expression in bracket is bounded for all $\alpha$, $\beta$ and $n$. However if $\alpha \to \beta$, i.e, $\alpha - \beta \to 0$; then for large $n$ we get $\longrightarrow$

$$A = \lim_{n\to\infty} \frac{a}{n}\left[\text{Some finite quantity}\right] + \lim_{n\to\infty} \frac{a}{n} \cdot \frac{\sin \frac{n(\alpha-\beta)}{2} \cdot \sin \frac{(n+1)(\alpha-\beta)}{2}}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow A = 0 + a \sin \frac{\{(n+1)(\alpha-\beta)\}}{2} \text{ for large } n \quad \left[\because \lim_{\theta\to 0} \frac{\sin n\theta}{\sin\theta} = n\right]$$

$$\Rightarrow A = a \sin \frac{(n+1)(\alpha-\beta)}{2}$$

Thus, for large $n$,

if $\alpha \neq \beta \Rightarrow \lambda \neq \mu$, then $A \to 0$ and

if, $\alpha \longrightarrow \beta \Rightarrow \lambda \to \mu$, then $A \to a \sin \frac{(n+1)(\alpha-\beta)}{2}$

Now,

$$B = \frac{2}{n} \sum_{t=1}^{n} y_t \sin\left(\frac{2\pi}{\mu} \cdot t\right)$$

$$\Rightarrow B = \frac{2}{n} \sum_{t=1}^{n} \left\{a \sin\left(\frac{2\pi}{\lambda}\right) . t + \varepsilon_t\right\} . \sin\left(\frac{2\pi}{\mu}\right) . t \quad [\text{from (i)}]$$

$$\Rightarrow B = \frac{a}{n} \sum_{t=1}^{n} 2 \sin\left\{\left(\frac{2\pi}{\lambda}\right) t\right\} . \sin\left\{\left(\frac{2\pi}{\mu}\right) . t\right\} \quad [\because \varepsilon_t \text{ is un-correlated}]$$

$$\Rightarrow B = \frac{a}{n} \sum_{t=1}^{n} 2 \sin \alpha t \sin \beta t \quad \left[\text{let } \alpha = \frac{2\pi}{\lambda}, \beta = \frac{2\pi}{\mu}\right]$$

$$\Rightarrow B = \frac{a}{n} \sum_{t=1}^{n} \{\cos(\alpha-\beta)t - \cos(\alpha+\beta)t\} \quad \left[\text{using, } 2\sin a \sin b = \cos(a-b) - \cos(a+b)\right] \longrightarrow (v)$$

Let,

$$S = \sum_{t=1}^{n} \cos(\alpha - \beta)t$$

$$\Rightarrow S.\sin\frac{(\alpha-\beta)}{2} = \sum_{t=1}^{n} \cos(\alpha-\beta).t \quad . \sin\frac{(\alpha-\beta)}{2}$$

$$\Rightarrow S.\sin\frac{(\alpha-\beta)}{2} = \frac{1}{2}\sum_{t=1}^{n} 2\left[\cos\{(\alpha-\beta).t\}.\sin\frac{(\alpha-\beta)}{2}\right]$$

$$\Rightarrow S.\sin\frac{(\alpha-\beta)}{2} = \frac{1}{2}\sum_{t=1}^{n}\left[\sin\left\{(\alpha-\beta)t + \frac{(\alpha-\beta)}{2}\right\} - \sin\left\{(\alpha-\beta).t - \frac{(\alpha-\beta)}{2}\right\}\right]$$

$[2\cos a \sin b = \sin(a+b) - \sin(a-b)]$

$$\Rightarrow S.\sin\frac{(\alpha-\beta)}{2} = \frac{1}{2}\left[\left\{\sin\frac{3(\alpha-\beta)}{2} - \sin\frac{(\alpha-\beta)}{2}\right\} + \left\{\sin\frac{5(\alpha-\beta)}{2} - \sin\frac{3(\alpha-\beta)}{2}\right\} + \dots + \left\{\sin\frac{(2n+1)(\alpha-\beta)}{2} - \sin\frac{(2n-1)(\alpha-\beta)}{2}\right\}\right]$$

$$\Rightarrow S.\sin\frac{(\alpha-\beta)}{2} = \frac{1}{2}(-)\left[\sin\frac{(\alpha-\beta)}{2} - \sin\frac{(2n+1)(\alpha-\beta)}{2}\right]$$

$$\Rightarrow S.\sin\frac{(\alpha-\beta)}{2} = -\frac{1}{\cancel{2}}.\cancel{2}\cos\frac{2(n+1)(\alpha-\beta)}{2.2} \sin\frac{-2n(\alpha-\beta)}{2}$$

$$\Rightarrow S.\sin\frac{(\alpha-\beta)}{2} = \cos.\frac{(n+1)(\alpha-\beta)}{2}\sin\frac{n(\alpha-\beta)}{2}$$

$$\Rightarrow S = \frac{\cos\frac{(n+1)(\alpha-\beta)}{2}\sin\frac{n(\alpha-\beta)}{2}}{\sin\frac{(\alpha-\beta)}{2}} \quad \text{———(vi)}$$

Again,

$$S = \sum_{t=1}^{n} \cos(\alpha+\beta)t$$

$$\Rightarrow S.\sin\frac{(\alpha+\beta)}{2} = \sum_{t=1}^{n}\cos(\alpha+\beta).t \quad . \sin\left(\frac{\alpha+\beta}{2}\right)$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\sum_{t=1}^{n} 2\cos(\alpha+\beta).t \sin\left(\frac{\alpha+\beta}{2}\right)$$

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$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = \frac{1}{2}\sum_{t=1}^{n}\left[\sin\left\{(\alpha+\beta).t + \frac{(\alpha+\beta)}{2}\right\} - \sin\left\{(\alpha+\beta)t - \frac{(\alpha+\beta)}{2}\right\}\right]$$

$[2\cos a \sin b = \sin(a+b) - \sin(a-b)]$

$$\Rightarrow S.\sin\frac{(\alpha+\beta)}{2} = \frac{1}{2}\left[\left\{\sin\frac{3(\alpha+\beta)}{2} - \sin\frac{(\alpha+\beta)}{2}\right\} + \left\{\sin\frac{5(\alpha+\beta)}{2} - \sin\frac{3(\alpha+\beta)}{2}\right\} + \dots + \left\{\sin\frac{(2n+1)(\alpha+\beta)}{2} - \sin\frac{(2n-1)(\alpha+\beta)}{2}\right\}\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = -\frac{1}{2}\left[\sin\frac{(\alpha+\beta)}{2} - \sin\frac{(2n+1)(\alpha+\beta)}{2}\right]$$

$$\Rightarrow S.\sin\left(\frac{\alpha+\beta}{2}\right) = -\frac{1}{\cancel{2}}.\cancel{2}\cos\frac{\cancel{2}(n+1)(\alpha+\beta)}{\cancel{2}.2}\sin\frac{-\cancel{2}n(\alpha+\beta)}{\cancel{2}}$$

$\left[\sin a - \sin b = 2\cos\frac{a+b}{2}\sin\frac{a-b}{2}\right]$

$$\Rightarrow S = \frac{\sin\frac{n(\alpha+\beta)}{2}\cos\frac{(n+1)(\alpha+\beta)}{2}}{\sin\frac{(\alpha+\beta)}{2}} \quad \text{——(vii)}$$

$\therefore$ From eqn (v), (vi) and (vii) $\rightarrow$

$$B = \frac{a}{n}\left[\frac{\cos\frac{(n+1)(\alpha-\beta)}{2}\sin\frac{n(\alpha-\beta)}{2}}{\sin\left(\frac{\alpha-\beta}{2}\right)} + \frac{\sin\frac{n(\alpha+\beta)}{2}\cos\frac{(n+1)(\alpha+\beta)}{2}}{\sin\frac{(\alpha+\beta)}{2}}\right]$$

If $\alpha \neq \beta$, then $B \to 0$ for large $n$, since then the expression in bracket is bounded for all $\alpha$, $\beta$ and $n$. However if $\alpha \to \beta$, i.e; $\alpha - \beta \to 0$, then for large $n$, we get $\longrightarrow$

$$B = \lim_{n\to\infty}\frac{a}{n}\left[\text{Some finite quantity}\right] + \lim_{n\to\infty}\frac{a}{n}\frac{\cos\frac{(n+1).(\alpha-\beta)}{2}\sin\frac{n(\alpha-\beta)}{2}}{\sin\frac{(\alpha-\beta)}{2}}$$

$$\Rightarrow B = a\cos\frac{(n+1)(\alpha-\beta)}{2} \qquad \left[\because \lim_{\theta\to 0}\frac{\sin n\theta}{\sin\theta} = n\right]$$

Thus, for large $n$,

if $\alpha \neq \beta \Rightarrow \lambda \neq \mu$, then $B \to 0$

and if, $\alpha \to \beta \Rightarrow \lambda \to \mu$, then $B \to a \cos \frac{(n+1)(\alpha-\beta)}{2}$

Thus, if the arbitrary number '$\mu$' is exactly the period of os ($\lambda$) of the series, then

$$S^2(\mu) = a^2$$

$$\Rightarrow S(\mu) = a$$

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We now take a number of trial period $\mu$ round about the true period $\lambda$, which may be guessed by plotting the data on a graph paper, and calculate $R_\mu^2$ in each case. Finally, we draw a graph plotting $R_\mu^2$ against $\mu$. The diagram, called a *periodogram*, is a simple device for finding the true cyclical period $\lambda$ in a time series by equating it to that value of $\mu$ for which $R_\mu^2$ attains a maximum.

Similarly, if the cyclical component is composed of several periodic terms, say with periods $\lambda_1$, $\lambda_2$, ......, $\lambda_k$, $R_\mu^2$ will remain small unless the trial period $\mu$ coincides with one of the true periods, in which case it attains a local maximum with value equal to the square of the amplitude of the periodic term concerned. This is shown in the figure below.

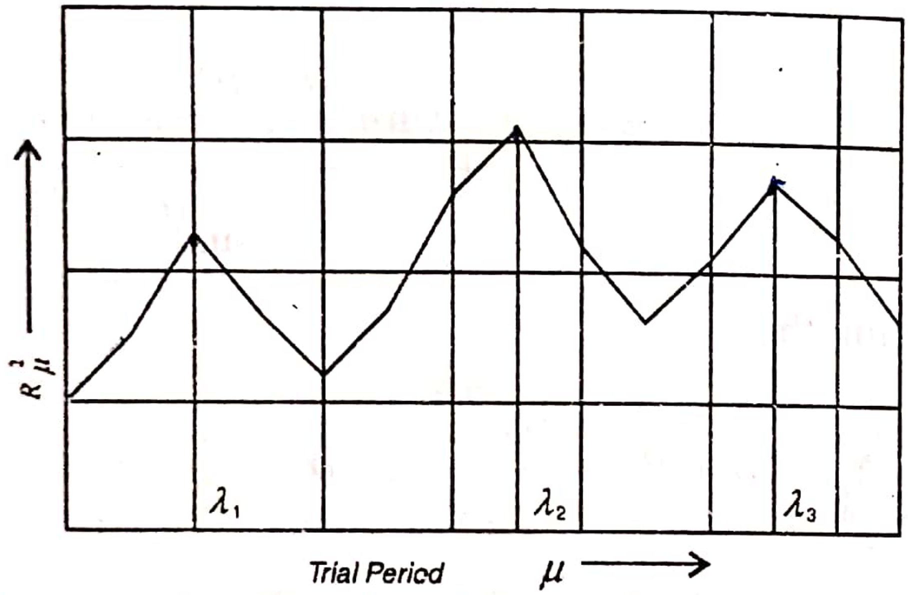


Fig. 7.11 A typical periodogram.