Q) Random sample of equal sizes i.e., $n_1 = n_2 = n$ drawn from two independent normal populations $N(\mu_1, \sigma_1^2)$ and $N(\mu_2; \sigma_2^2)$, where μ_1 and μ_2 both are unknown also σ_1^2 and σ_2^2 are also unknown obtain $100(1-\alpha)\%$ confidence interval for the parametric function $\Psi = \mu_1/\mu_2$.

Proof:

Consider the random variables $U = X_1 - \Psi(X_2)$, where X_1 is a $N(\mu_1, \sigma_1^2)$ and X_2 is another $N(\mu_2, \sigma_2^2)$ variate. Then expectation of U,

$$E(U) = E(X_1) - \Psi E(X_2)$$

$$= \mu_1 - \frac{\mu_1}{\mu_2}\mu_2$$

$$= 0$$

and $$V(U) = V(X_1) + \Psi^2 V(X_2) - 2\Psi \operatorname{cov}(X_1, X_2)$$

$$= \cancel{\sigma_1^2 + \frac{\mu_1^2}{\mu_2^2}\sigma_2^2}$$

$$= V(X_1) + \Psi^2 V(X_2) \quad [\because X_1 \text{ and } X_2 \text{ are independent}]$$

$$\sigma_U^2 = \sigma_1^2 + \Psi^2 \sigma_2^2$$

$$\hat{\sigma}_u^2 = S_U^2 = S_1^2 + \Psi S_2^2$$

where, $S_1^2 = \frac{1}{n_1 - 1}\sum(x_{1i} - \bar{x}_1)^2$

and $S_2^2 = \frac{1}{n_2 - 1}\sum(x_{2i} - \bar{x}_2)^2$

$\therefore \bar{U} = \bar{x}_1 - \psi \bar{x}_2$

$$E(\bar{U}) = E(\bar{x}_1) - \psi E(\bar{x}_2)$$

$$= \mu_1 - \frac{\mu_1}{\mu_2} \cdot \mu_2$$

$$= 0.$$

$$V(\bar{U}) = V(\bar{x}_1) + \psi^2 V(\bar{x}_2)$$

$$= \frac{\sigma_1^2}{n_1} + \psi^2 \frac{\sigma_2^2}{n_2}$$

$$= \frac{\sigma_1^2 + \psi^2 \sigma_2^2}{n}$$

$\therefore \bar{U} = \bar{x}_1 - \psi \bar{x}_2$ is a $N\left(0, \frac{\sigma_1^2 + \psi^2 \sigma_2^2}{n}\right)$.

Since σ_1^2 and σ_2^2 are not known, the statistic

$$g(T, \theta) = \frac{\bar{x}_1 - \psi \bar{x}_2}{s_u / \sqrt{n}} \sim t \text{ with } (n-1) \text{ d.f}$$

$$\therefore P_\theta \left[\left| \frac{\bar{x}_1 - \psi \bar{x}_2}{s_u / \sqrt{n}} \right| \le t_{\alpha/2, n-1} \right] = 1 - \alpha$$

$$\text{or, } P_\theta \left[(\bar{x}_1 - \psi \bar{x}_2)^2 \le t^2_{\alpha/2, n-1} \cdot \frac{s_u^2}{n} \right] = 1 - \alpha$$

$$\text{or, } P_\theta \left[(\bar{x}_1 - \psi \bar{x}_2)^2 \le t^2_{\alpha/2, n-1} \cdot \frac{s_1^2 + \psi s_2^2}{n} \right] = 1 - \alpha$$

or, $P_\theta\left[\bar{x}_1^2 - 2\psi\bar{x}_1\bar{x}_2 + \psi^2\bar{x}_2^2 \leq t^2_{\alpha/2, n-1}\dfrac{s_1^2+\psi^2 s_2^2}{n}\right] = 1-\alpha$

or, $P_\theta\left[\bar{x}_1^2 - 2\psi\bar{x}_1\bar{x}_2 + \psi^2\bar{x}_2^2 - t^2_{\alpha/2, n-1}\cdot\dfrac{s_1^2}{n} - t^2_{\alpha/2, n-1}\cdot\dfrac{\psi^2 s_2^2}{n} \leq 0\right]$

$= 1-\alpha$

or, $P_\theta\left[\psi^2\left(\bar{x}_2^2 - \dfrac{t^2_{\alpha/2, n-1}\cdot s_2^2}{n}\right) - 2\psi\bar{x}_1\bar{x}_2 + \bar{x}_1^2 - t^2_{\alpha/2, n-1}\cdot\dfrac{s_1^2}{n} \leq 0\right]$

$= 1-\alpha$

If θ_1 and θ_2 be the roots $(\theta_2 > \theta_1)$ of the quadratic equation in ψ,

$$\psi^2\left(\bar{x}_2^2 - \frac{t^2_{\alpha/2, n-1}\cdot s_2^2}{n}\right) - 2\psi\bar{x}_1\bar{x}_2 + \bar{x}_1^2 - t^2_{\alpha/2, n-1}\cdot\frac{s_1^2}{n} = 0$$

then $100(1-\alpha)\%$ confidence limits of $\psi = \mu_1/\mu_2$ will be between θ_1 and θ_2.

Confidence Interval of Population Proportion

Confidence interval of single proportion:

Let, $x_1, x_2, \ldots, x_n$ be the random samples of size n drawn from the bernoullian population with parameter P.

Let,

x be the statistic (= no. of units ~~belonging~~ belong to a particular characteristic) defined for the purpose.

$\therefore x \sim B(n, P)$

Now, $E(x) = nP$ and $V(x) = nP(1-P)$

Then, x/n is the sample proportion of that characteristic $= p$ (say)

Therefore, $E(p) = E(x/n) = P$

and $V(p) = 1/n^2 \, V(x)$

$$= \frac{P(1-P)}{n}$$

Under large sample normality assumption

$p \sim N\left(P, \frac{P(1-P)}{n}\right)$,

$$\therefore \psi(p, P) = \frac{p - P}{\sqrt{\frac{P(1-P)}{n}}} \sim N(0,1)$$

$\therefore$ We have $P_\theta\left[-z_{\alpha/2} \leq \frac{p - P}{\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}} \leq +z_{\alpha/2}\right] = 1-\alpha$

or, $P_\theta\left[-z_{\alpha/2}\cdot\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} \leq p - P \leq +z_{\alpha/2}\cdot\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}\right] = 1-\alpha$

or, $P_\theta\left[-\left(p + z_{\alpha/2}\cdot\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}\right) \leq P \leq -\left(p - z_{\alpha/2}\cdot\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}\right)\right] = 1-\alpha$

or, $P_\theta\left[p - z_{\alpha/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} \leq P \leq p + z_{\alpha/2}\sqrt{\frac{\hat{P}(1-\hat{P})}{n}}\right] = 1-\alpha$

Hence $100(1-\alpha)\%$ confidence limits for P will be $p \pm z_{\alpha/2}\cdot\sqrt{\frac{\hat{P}(1-\hat{P})}{n}} = p \pm z_{\alpha/2}\cdot\sqrt{\frac{p(1-p)}{n}}$