Confidence interval for parameter for some distributions not belonging to the location - scale family

Theorem 1:

Let $F(x, \theta)$ be the cumulative distribution function of a random variable X. Then $X(x, \theta) \sim U(0, 1)$.

Theorem 2:

If $X \sim U(0,1)$ then $-\log(x) \sim \exp(1)$

Theorem 3:

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution from a density function

$$f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}; \quad 0 < x < \infty, \theta > 0$$
$$= 0 \quad ; \text{ otherwise}$$

where $\theta (> 0)$ is a paramete.

Then the r.v $2/\theta \sum_{i=1}^{n} x_i \sim \chi^2_{2n}$ d.f.

Proof:

Let $Y = 2/\theta \sum_{i=1}^{n} X_i$.

Now, we show that the sampling distribution of Y is χ^2 variate with $2n$ d.f.

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we use the m.g.f method to show this,

The moment generating function of Y is given by

$$M_Y(t) = M_{\frac{2}{\theta}\Sigma x_i}(t)$$

$$= M_{\Sigma x_i}\left(\frac{2}{\theta}t\right)$$

$$= M_{X_1}\left(\frac{2}{\theta}t\right) \cdot M_{X_2}\left(\frac{2}{\theta}t\right) \cdots M_{X_n}\left(\frac{2}{\theta}t\right)$$

$$= \prod_{i=1}^{n} M_{X_i}\left(\frac{2}{\theta}t\right)$$

$$= \prod_{i=1}^{n}\left(1 - \theta\frac{2}{\theta}t\right)^{-1}$$

$$= (1-2t)^{-n}$$

$$= (1-2t)^{-\frac{2n}{2}}$$

∴ which looks like the M.G.F of a χ^2 variate. ~~with 2n d.f~~.

Hence by uniqueness theorem of m.g.f.

$\frac{2}{\theta}\Sigma x_i \sim \chi^2_{2n}$.

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Q) Let $x_1, x_2, \ldots, x_n$ be a random sample from a distribution with density function $f(x,\theta) = \theta x^{(\theta-1)}, 0 \leq x \leq 1$

where, $\theta(>0)$ is a parameter. Then the r.v $-2\theta \sum_{i=1}^{n} \ln(x_i)$ has a χ^2 distribution with $2n$ d.f.

Soln We are given that, $x_i \sim \theta x^{\theta-1}$;

Hence, c.d.f is $F(x,\theta) = \int_0^x \theta t^{(\theta-1)} dt$

$$= \theta \int_0^x t^{(\theta-1)} dt$$

$$= \cancel{\theta \left[\frac{t^{\theta-1}}{\log} \right]_0^x}$$

$$= \theta \left[\frac{t^\theta}{\theta} \right]_0^x$$

$$= x^\theta.$$

$\therefore F(x_i,\theta) \sim U(0,1)$

i.e., $x_i^\theta \sim U(0,1)$

$\therefore -\ln(x_i^\theta) \sim \exp(1)$

i.e., $-\theta \ln(x_i) \sim \exp(1)$

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Now

$$M_{-2\theta\sum \ln(x_i)}(t)$$

$$= M_{-\theta\sum \ln(x_i)}(2t)$$

$$= M_{-\sum \theta \ln(x_i)}(2t)$$

$$= \prod_{i=1}^{n} M_{X_i}(2t)$$

$$= (1-2t)^{-n}$$

Let,

$$-\theta \ln(x_i) = X_i$$

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$\therefore -2\theta \sum \ln(x_i) \sim \chi^2_{2n}$

Q) If $X_1, X_2, \ldots, X_n$ is a random sample from a population with density ~~f(x)~~

$f(x;\theta) = \theta x^{\theta-1}; 0 < x < 1$

where $\theta (>0)$ is an unknown parameter, what is a $100(1-\alpha)\%$ confidence interval for θ.

Soln: To construct a confidence interval for θ we need a pivotal quantity. That is we need a random variable which is a function of the sample and the parameter and whose probability distribution is known but does not involve θ. We use the random variable $U = -2\theta \sum \ln(x_i) \sim \chi^2_{2n}$ as the pivotal quantity.

The $100(1-\alpha)\%$ confidence interval for θ can be constructed from

$$P_\theta\left[\chi^2_{\alpha/2, 2n} \le -2\theta \sum \ln(x_i) \le \chi^2_{(1-\alpha/2), 2n}\right] = 1-\alpha$$

or, $$P_\theta\left[\chi^2_{\alpha/2, 2n} / -2\sum \ln(x_i) \le \theta \le \chi^2_{(1-\alpha/2), 2n} / -2\sum \ln(x_i)\right] = 1-\alpha$$

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Hence $100(1-\alpha)\%$ confidence interval for θ is given by-

$$\left[\chi^2_{\alpha/2, 2n} \Big/ -2\sum \ln(x_i) \; , \; \chi^2_{(1-\alpha/2), 2n} \Big/ -2\sum \ln(x_i) \right]$$

<u>Theorem</u>

Let, $X_1, X_2, \ldots, X_n$ be a random sample from a continuous population X with a d.f $F(x, \theta)$. If $F(x, \theta)$ is monotone in θ, then the statistic $Q = -2\sum_{i=1}^{n} \ln F(x_i, \theta)$ is a pivotal quantity and has a χ^2 distribution with $2n$ d.f. It should be noted that the condition $F(x, \theta)$ is monotone in θ is needed to ensure an interval. Otherwise we may get a confidence region instead of a confidence interval. Further note that the statistic $-2\sum_{i=1}^{n} \ln(1 - F(x_i, \theta))$ is also has a χ^2 distribution with $2n$ d.f, i.e.

$$-2\sum \ln(1 - F(x_i, \theta)) \sim \chi^2_{2n}.$$

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Q) If $x_1, x_2, \ldots x_n$ is a random sample from a distribution with density function

$$f(x, \theta) = \frac{1}{\theta} \; ; \text{if } 0 < x < \theta$$
$$= 0 \; ; \text{otherwise.}$$

where $\theta (>0)$ is a parameter. Then what is the $100(1-\alpha)\%$ confidence interval of θ.

<u>Soln</u>

The cumulative density function of $f(x, \theta)$ is,

$$F(x, \theta) = \int_0^x \frac{1}{\theta} dt$$
$$= x/\theta .$$

Since $-2 \sum \ln F(x_i, \theta) \sim \chi^2_{2n}$

$\therefore$ $100(1-\alpha)\%$ confidence interval for θ can be constructed from

$$P_\theta \left[\chi^2_{\alpha/2, 2n} \leq -2 \sum \ln F(x_i, \theta) \leq \chi^2_{(1-\alpha/2), 2n} \right] = 1 - \alpha .$$

Now, $-2 \sum \ln F(x_i, \theta) = -2 \sum \ln (x_i/\theta)$

$$= -2 \sum \{ \ln(x_i) - \ln(\theta) \}$$
$$= -2 \sum \ln x_i + 2 \sum \ln \theta$$
$$= 2n \ln(\theta) - 2 \sum \ln(x_i)$$

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$$\therefore P_\theta\left[\chi^2_{\alpha/2,2n} \le 2n\ln(\theta) - 2\sum\ln(x_i) \le \chi^2_{(1-\alpha/2),2n}\right] = 1-\alpha$$

$$\text{or, } P_\theta\left[\chi^2_{\alpha/2,2n} + 2\sum\ln(x_i) \le 2n\ln(\theta) \le \chi^2_{(1-\alpha/2),2n} + 2\sum\ln(x_i)\right] = 1-\alpha$$

$$\text{or, } P_\theta\left[e^{\left(\frac{\chi^2_{\alpha/2,2n} + 2\sum\ln(x_i)}{2n}\right)} \le \theta \le e^{\left(\frac{\chi^2_{(1-\alpha/2),2n} + 2\sum\ln(x_i)}{2n}\right)}\right] = 1-\alpha$$

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Q) If $x_1, x_2, \ldots, x_n$ is a random sample from a distribution with density function

$f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}$; $0 < x < \infty$.

where $\theta (>0)$ is a parameter. Then what is the $100(1-\alpha)\%$ confidence interval for θ.

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$$f(x, \theta) = \frac{1}{\theta} e^{-x/\theta}$$

$$\therefore F(x, \theta) = \int_0^x \frac{1}{\theta} e^{-t/\theta} dt$$

$$= \frac{1}{\theta} \left[\frac{e^{-t/\theta}}{-1/\theta} \right]_0^x$$

$$= 1 - e^{-x/\theta}$$

$$= 1 - F(x, \theta) = e^{-x/\theta}$$

$$-2 \sum \ln(1 - F(x_i, \theta))$$

$$= -2 \sum (-x/\theta)$$

$$= \frac{2}{\theta} \sum x_i$$

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$$\therefore P_\theta\left[\chi^2_{\alpha/2, 2n} \le -2\sum \ln(1-F(x_i,\theta)) \le \chi^2_{(1-\alpha/2), 2n}\right] = 1-\alpha$$

$$\text{or, } P_\theta\left[\chi^2_{\alpha/2, 2n} \le \frac{2}{\theta}\sum x_i \le \chi^2_{(1-\alpha/2), 2n}\right] = 1-\alpha$$

$$\text{or, } P_\theta\left[\frac{2\sum x_i}{\chi^2_{(1-\alpha/2), 2n}} \le \theta \le \frac{2\sum x_i}{\chi^2_{\alpha/2, 2n}}\right] = 1-\alpha.$$

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