Q) Confidence Interval using chebyshev's Inequality:

Let, $x_1, x_2, \ldots, x_n$ be a random sample of n observations from a population having $f(x, \theta); \theta \in \Theta$. Consider that

$$T = t(x_1, x_2, x_3, \ldots, x_n)$$

is a sample statistic and it ~~has an~~ is an unbiased estimator of θ $(E(T) = \theta)$. Then by Chebyshev's inequality we can write

$$P\left[|T - \theta| \leq \epsilon\right] \geq 1 - \frac{Var(T)}{\epsilon^2}$$

Let, $\epsilon = k\sqrt{Var(T)}$

then, $P\left[|T-\theta| \leq k\sqrt{Var(T)}\right] \geq 1 - \frac{1}{k^2}$

again, let $1 - \alpha = 1 - \frac{1}{k^2}$

$$\Rightarrow k = \frac{1}{\sqrt{\alpha}}$$

~~then, $P[T-\sqrt{Var(T)}$~~

then,

$$P\left[|T-\theta| \leq k\sqrt{V(T)}\right] \geq 1-\alpha$$

or, $$P\left[-k\sqrt{V(T)} \leq T-\theta \leq +k\sqrt{V(T)}\right] \geq 1-\alpha.$$

or, $$P\left[-(T+k\sqrt{V(T)}) \leq -\theta \leq -(T-k\sqrt{V(T)})\right] \geq 1-\alpha$$

or, $$P\left[-\left(T+\sqrt{\frac{V(T)}{\alpha}}\right) \leq -\theta \leq -\left(T-\sqrt{\frac{V(T)}{\alpha}}\right)\right] \geq 1-\alpha$$

or, $$P\left[T-\sqrt{\frac{V(T)}{\alpha}} \leq \theta \leq T+\sqrt{\frac{V(T)}{\alpha}}\right] \geq 1-\alpha.$$

Hence, the $100(1-\alpha)\%$ confidence interval of θ is $\left[T-\sqrt{\frac{V(T)}{\alpha}},\ T+\sqrt{\frac{V(T)}{\alpha}}\right]$.

Q) Let, $x_1, x_2, \ldots, x_n$ be a random sample of n observations from a population with p.d.f,

$$f(x) = 1/\theta \ ; \ 0 < x < \theta$$

Find $100(1-\alpha)\%$ confidence interval for θ using chebyshev's inequality.

Soln.

$$f(x) = \frac{1}{\theta} \; ; \; 0 < x < \theta$$

$$E(X) = \int_0^\theta x \frac{1}{\theta} dx$$

$$= \frac{1}{\theta}\left[\frac{x^2}{2}\right]_0^\theta$$

$$= \frac{\theta}{2}$$

$\therefore E(X) = \theta/2$

$\therefore E(2X) = \theta$

$\therefore T = 2X$ is the unbiased estimator of θ.

Now,

$$V(T) = 4\,V(X)$$

Again, $V(X) = E(X^2) - \{E(X)\}^2$

$$= \int_0^\theta x^2 \frac{1}{\theta} dx - \frac{\theta^2}{4}$$

$$= \frac{1}{\theta}\left[\frac{x^3}{3}\right]_0^\theta - \frac{\theta^2}{4}$$

$$= \frac{\theta^2}{3} - \frac{\theta^2}{4}$$

$$= \theta^2/12$$

$$\therefore V(T) = 4 \cdot \theta^2/12 = \frac{\theta^2}{3}$$

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The $100(1-\alpha)\%$ confidence interval for θ

By Chebyshev's inequality,

$$\cancel{P\left\{T-\sqrt{\frac{V(T)}{\alpha}},\ T+\sqrt{\frac{V(T)}{\alpha}}\right\}}$$

$$\cancel{\{2x-}$$

$$P\left[T-\sqrt{\frac{V(T)}{\alpha}} \le \theta \le T+\sqrt{\frac{V(T)}{\alpha}}\right]$$

$$= P\left[2x-\sqrt{\frac{\theta^2}{3\alpha}} \le \theta \le 2x+\sqrt{\frac{\theta^2}{3\alpha}}\right]$$

Now,

$$2x-\sqrt{\frac{\theta^2}{3\alpha}} \le \theta \qquad\qquad \theta \le 2x+\sqrt{\frac{\theta^2}{3\alpha}}$$

$$\Rightarrow 2x \le \theta + \frac{\theta}{\sqrt{3\alpha}} \qquad\qquad \Rightarrow \theta - \theta\sqrt{\frac{1}{3\alpha}} \le 2x$$

$$\Rightarrow 2x \le \theta\left(1+\frac{1}{\sqrt{3\alpha}}\right) \qquad\qquad \Rightarrow \theta\left(1-\frac{1}{\sqrt{3\alpha}}\right) \le 2x$$

$$\Rightarrow \frac{2x}{1+\frac{1}{\sqrt{3\alpha}}} \le \theta \qquad\qquad \Rightarrow \theta \le \frac{2x}{1-\frac{1}{\sqrt{3\alpha}}}$$

Hence, the $100(1-\alpha)\%$ confidence interval is

$$\left[\frac{2x}{1+\frac{1}{\sqrt{3\alpha}}},\ \frac{2x}{1-\frac{1}{\sqrt{3\alpha}}}\right]$$