Multinomial distribution

Let $X_1, X_2, \cdots X_K$ be K jointly distributed random variables, each of which is discrete and integer-value such that

$$\sum_{i=1}^{K} X_i = m,$$

a positive integer. Suppose the joint probability-mass function of $X_1, X_2, \cdots, X_K$ is

$$f(x_1, x_2, \cdots x_K) = \binom{m}{x_1, x_2, \cdots x_K} p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K}$$

where $\sum_{i=1}^{K} x_i = m$ and $\sum_{i=1}^{K} p_i = 1$

and $p_i > 0$ for $i = 1, 2, \cdots K$

$= 0$, elsewhere.

Here $p_1, p_2, \cdots p_K$ are all parameters and $X_1, X_2, \cdots X_K$ are called multinomial variables.

M.G.F of multinomial distribution

The m.g.f of multinomial distribution is given by.

$$M(t_1, t_2, \cdots t_K) = E\left[e^{\sum_{i=1}^{K} t_i X_i}\right]$$

$$= \sum e^{\sum_{i=1}^{K} t_i x_i} \frac{m!}{x_1!\, x_2! \cdots x_K!} p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K}$$

$$= \sum \frac{m!}{x_1!\, x_2! \cdots x_K!} (p_1 e^{t_1})^{x_1} (p_2 e^{t_2})^{x_2} \cdots (p_K e^{t_K})^{x_K}$$

$$= (p_1 e^{t_1} + p_2 e^{t_2} + \cdots + p_K e^{t_K})^{m}$$

Now, $\mu_i' = E(X_i)$

$$= \left(\frac{\partial M}{\partial t_i}\right)_{t_i=0}$$

$$= m(p_1 e^{t_1} + \cdots + p_k e^{t_k})^{m-1} p_i e^{t_i}\Big]_{t_i=0}$$

$$= m p_i$$

$$E(X_i^2) = \left(\frac{\partial^2 M}{\partial t_i^2}\right)_{t_i=0}$$

$$= m(m-1) p_i^2 e^{2t_i} (p_1 e^{t_1} + \cdots + p_k e^{t_k})^{m-2}$$

$$+ m(p_1 e^{t_1} + \cdots + p_k e^{t_k})^{m-1} p_i e^{t_i}\Big]_{t_i}$$

$$= m(m-1) p_i^2 + m p_i$$

$\therefore Var(X_i) = \sigma_i^2$

$$= m(m-1) p_i^2 + m p_i - m^2 p_i^2$$

$$= m p_i - m p_i^2$$

$$= m p_i (1 - p_i).$$

Now, $E(X_i X_j) = m(m-1) p_i p_j e^{(t_i+t_j)} (p_1 e^{t_1} + \cdots + p_k e^{t_k})$

$$= m(m-1) p_i p_j.$$

$\therefore Cov(X_i X_j) = E(X_i X_j) - E(X_i) E(X_j)$

$$= m(m-1) p_i p_j - m^2 p_i p_j$$

$$= - m p_i p_j$$

It follow that the correlation coefficient between X_i ...
is $\rho = \dfrac{-m p_i p_j}{\sqrt{m p_i(1-p_i)}\sqrt{m p_j(1-p_j)}} = -\sqrt{\dfrac{p_i p_j}{(1-p_i)(1-p_j)}}$

Marginal distribution :-

The marginal distⁿ of X_1 is obtain by summing over other $(K-1)$ variables $X_2, X_3, \cdots, X_K$ as.

$$P(X_1 = x_1) = \sum_{x_2}{}' \sum_{x_3}{}' \cdots \sum_{x_K}{}' \frac{m!}{x_1!\, x_2! \cdots x_K!} p_1^{x_1} p_2^{x_2} \cdots p_K^{x_K}$$

$$= \frac{m!\, p_1^{x_1}}{x_1!\,(m-x_1)!} \sum_{x_2}{}' \sum_{x_3}{}' \cdots \sum_{x_K}{}' \frac{(m-x_1)!}{x_2!\, x_3! \cdots x_K!} p_2^{x_2} \cdots p_K^{x_K}$$

$$= \binom{m}{x_1} p_1^{x_1} (p_2 + p_3 + \cdots + p_K)^{m-x_1}$$

$$= \binom{m}{x_1} p_1^{x_1} (1-p_1)^{m-x_1}$$

This means that the marginal distribution of X_1 is of the binomial type with parameters m and p_1

Proceeding in the similar way the marginal distⁿ of any q $(q < K-1)$ of the variables say $X_1, X_2, \cdots, X_p$ can be obtain as

$$P(X_1 = x_1, \cdots, X_q = x_q) = \frac{m!}{x_1!\, x_2! \cdots x_q!\,(m - x_1 - x_2 - \cdots - x_q)!} \times p_1^{x_1} p_2^{x_2} \cdots (1 - p_1 - p_2 - \cdots - p_q)^{m - \sum_{i=1}^{q} x_i}$$

which is also of the multinomial type with paramete m and $p_1, p_2, \cdots, p_q$

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Conditional distribution:

The conditional dist^n of $X_1 = x_1$ given $X_2 = x_2 \cdots X_{K-1} = x_{K-1}$ is obtain as.

$$P(X_1 = x_1 / X_2 = x_2 \cdots X_{K-1} = x_{K-1}) = \frac{P(X_1 = x_1, X_2 = x_2 \cdots, X_{K-1} = x_{K-1})}{P(X_2 = x_2, X_3 = x_3 \cdots X_{K-1} = x_{K-1})}$$

$$= \frac{\dfrac{n!}{x_1!\, x_2! \cdots x_{K-1}!\, (n - x_1 - x_2 - \cdots - x_{K-1})!}\, p_1^{x_1} p_2^{x_2} \cdots (1 - p_1 - p_2 - \cdots - p_{K-1})^{n}}{\dfrac{n!}{x_2!\, x_3! \cdots x_{K-1}!\, (n - x_2 - x_3 - \cdots - x_{K-1})!}\, p_2^{x_2} p_3^{x_3} \cdots (1 - p_2 - p_3 - \cdots - p_{K-1})^{n}}$$

$$= \frac{(n - x_2 - x_3 - \cdots - x_{K-1})!}{x_1!\, (n - x_1 - x_2 - \cdots - x_{K-1})!} \left(\frac{p_1}{1 - p_2 - p_3 - \cdots - p_{K-1}}\right)^{x_1}$$

$$\times \left\{1 - \frac{p_1}{(1 - p_2 - p_3 - \cdots - p_{K-1})}\right.$$

Hence the conditional distribution of X_1 given X_2, X_3 is of the binomial type with parameters $(n - x_2 - x_3$

and $\dfrac{p_1}{1 - p_2 - p_3 - \cdots - p_{K-1}} = P$ (say)

Now the regression of X_1 on the others $X_2, X_3, \cdots X_K$ is given by

$$E(X_1 / X_2, X_3 \cdots X_{K-1}) = (n - x_2 - x_3 - \cdots - x_{K-1}) \frac{p_1}{(1 - p_2 - p_3}$$

$$= nP - x_2 P - x_3 P \cdots - x_{K-1}$$