Method of variation of Parameters

Given diff eqn is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x) \longrightarrow (1)$$

Reduced eqn is

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \longrightarrow (2)$$

Here $P(x), Q(x)$ need not be constants - what is essential here is to know the complementary function of (1) in advance. Then a particular soln of y_p of (1) can be obtained by the method of variation of parameters.

Let the general soln of (2) is

$$y = C_1 y_1(x) + C_2 y_2(x)$$

where C_1 and C_2 are constants, y_1 and y_2 are L.I. soln of (2).

we shall replace the constants c_1 and c_2 by unknown functions $v_1(x)$ and $v_2(x)$. our problem is to determine v_1 and v_2 so that

$$y_p = v_1 y_1 + v_2 y_2 \longrightarrow (3)$$

Satisfies (1). This will be a particular soln of (1). To determine v_1 and v_2 we require two eqn's.

Diff. (3) we get

$$Dy_p = (v_1 y_1' + v_2 y_2') + (v_1' y_1 + v_2' y_2) \rightarrow (4)$$

we shall choose v_1 and v_2 such that

$$v_1' y_1 + v_2' y_2 = 0 \longrightarrow (5)$$

Then (4) reduces to

$$Dy_p = v_1 y_1' + v_2 y_2' \longrightarrow (6)$$

$$D^2 y_p = (v_1 y_1'' + v_2 y_2'') + (v_1' y_1' + v_2' y_2') \rightarrow (7)$$

using (3), (6) and (7) in (1) we get,

$$v_1(y_1'' + Py_1' + Qy_1) + v_2(y_2'' + Py_2' + Qy_2) + v_1'y_1' + v_2'y_2' = R(x) \rightarrow (8)$$

Since y_1 and y_2 are soln of (2)

$$\therefore y_1'' + Py_1' + Qy_1 = 0 = y_2'' + Py_2' + Qy_2$$

P.T.O.

$\therefore$ (8) reduces to

$$v_1'y_1' + v_2'y_2' = R(x) \longrightarrow (9)$$

we solve (5) and (9) for v_1' & v_2'

$$v_1' = \frac{-y_2 R(x)}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \quad \text{and} \quad v_2' = \frac{y_1 R(x)}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}}$$

$$\therefore v_1 = -\int \frac{y_2 R(x)}{W(y_1, y_2)} dx \quad \text{and} \quad v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

$$\boxed{\therefore y_p = v_1(x) y_1(x) + v_2(x) y_2(x)}$$

Solve

1) $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$, $\boxed{y = c_1 \sin x + c_2 \cos x + \sin x \log(\sin x) - x\cos x}$

2) $\frac{d^2y}{dx^2} + a^2 y = \sec ax$, $y = c_1 \cos ax + c_2 \sin ax - \frac{1}{a^2} \cos ax (\log(\sec ax)) + \frac{1}{a} x \sin ax$

3) $(D^2 - 2D) y = e^x \cos x$, $y = c_1 + c_2 e^{2x} - \frac{1}{2} e^x \cos x$.

use variation of parameters to solve

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^{2x}$$

Soln :- Given differential eqn is

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^{2x}$$

or, $$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = e^{2x}$$

Here $P(x) = \frac{1}{x}$, $Q(x) = -\frac{1}{x^2}$, $R(x) = e^{2x}$

P.T.O.

Apply the method of variation of parameters to solve

1. $y_2 + n^2 y = \sec nx$, 2. $y_2 + y = \sec x$, 3. $y_2 + 4y = \sec 2x$

4. $y_2 + 9y = \sec 3x$, 5. $y_2 + a^2 y = \operatorname{cosec} ax$, 6. $y_2 + y = \operatorname{cosec} x$

7. $y_2 + 9y = \operatorname{cosec} 3x$, 8. $y_2 + 4y = 4\tan 2x$, 9. $y_2 + y = \tan x$.

10. $y_2 + a^2 y = \cot ax$, 11. $y_2 - y = \frac{2}{1+e^x}$, 12. $y_2 - 3y_1 + 2y = \frac{e^x}{1+e^x}$

13. $y_2 - 4y_1 + 3y = \frac{e^x}{1+e^x}$, 14. $y_2 - 2y_1 = e^x \sin x$,

15. $y_2 - 2y_1 + y = xe^x \sin x$, with $y(0) = 0$, and $(y_1)_{x=0} = 0$.

16. $x^2 y_2 + xy_1 - y = x^2 e^x$, 17. $x^2 y_2 + xy_1 - y = x$, Given that C.F. is $c_1 x + c_2 x^{-1}$.

18. $x^2 y_2 - xy_1 = x^3 e^x$, 19. $x^2 y_2 + 3xy_1 + y = \frac{1}{(1-x)^2}$

20. $x^2 y_2 + xy_1 - y = x^2 \log x$, $x > 0$.

Solve: $y_2 + 4y = 4\tan 2x$ using method of variation of parameters

Sol^n:- Given $y_2 + 4y = 4\tan 2x \dashrightarrow (1)$

<u>To find the C.F.</u>

Let $y = e^{mx} \neq 0$ be a trial solution.

$\therefore$ A.E. is $m^2+4=0$ ie $m = \pm 2i$

$\therefore$ C.F. $= C_1 \cos 2x + C_2 \sin 2x$

Let $y_1 = \cos 2x$, $y_2 = \sin 2x$

$$W(y_1, y_2) = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2(\cos^2 2x + \sin^2 2x) = 2 \neq 0$$

$\therefore$ $\cos 2x$ and $\sin 2x$ are L.I.

$\therefore$ The solution ~~cos 2x~~

Let $y_p = v_1 y_1 + v_2 y_2$, where v_1 & v_2 are functions of x.

Here $P(x) = 0$, $Q(x) = 4$, $R(x) = 4\tan 2x$.

So by using method of variation of parameters we have

$$v_1 = -\int \frac{y_2 R(x)}{W(y_1,y_2)}\,dx, \quad v_2 = \int \frac{y_1 R(x)}{W(y_1,y_2)}\,dx$$

$$v_1 = -\int \frac{\sin 2x \cdot 4\tan 2x}{2}\,dx$$

$$= -2\int \frac{\sin^2 2x}{\cos 2x}\,dx$$

$$= -2\int (\sec 2x - \cos 2x)\,dx$$

$$= -2\left\{\frac{\log|\sec 2x + \tan 2x|}{2} - \frac{\sin 2x}{2}\right\}$$

$$= \sin 2x - \log|\sec 2x + \tan 2x|$$

$$v_2 = \int \frac{\cos 2x \cdot 4\tan 2x}{2}\,dx$$

$$= 2\int \sin 2x\,dx$$

$$= -\cos 2x$$

$\therefore$ $y_p = \sin 2x \log|\sec 2x + \tan 2x| - \sin 2x \cos 2x$

$y_p = \sin 2x \cos 2x - \cos 2x \log|\sec 2x + \tan 2x|$

$= -\cos 2x \log|\sec 2x + \tan 2x|$

$\therefore$ The required solution is

$y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log|\sec 2x + \tan 2x|$

Ans

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Solve: $(D^2-2D)y = e^x \cos x$ by using method of variation of parameters

Solution: Given $(D^2-2D)y = e^x \cos x \longrightarrow (1)$

To find the C.F.

Let $y = e^{mx} \neq 0$ be a trial solution.

$\therefore$ A.E. is $m^2-2m=0$ i.e. $m(m-2)=0$ i.e. $m=0$ or, 2

$\therefore$ C.F. $= c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$

Let $y_1 = 1$ and $y_2 = e^{2x}$

Then Wronskian of y_1 & y_2 ie

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} \neq 0$$

$\therefore y_1 = 1$ and $y_2 = e^{2x}$ are linear independent. (L.I.)

Let $y_p = v_1 y_1 + v_2 y_2$, where v_1 & v_2 are functions of x.

Here $P(x) = -2$, $Q(x) = 0$, $R(x) = e^x \cos x$

So by using method of variation of parameters we have

$$v_1 = -\int \frac{y_2 R(x)}{W(y_1, y_2)} dx \quad \text{and} \quad v_2 = \int \frac{y_1 R(x)}{W(y_1, y_2)} dx$$

$$v_1 = -\int \frac{e^{2x} \cdot e^x \cos x}{2e^{2x}} dx = -\frac{1}{2}\int e^x \cos x \, dx = -\frac{1}{2} \times \frac{1}{2} e^x(\sin x + \cos x) = -\frac{1}{4} e^x(\sin x + \cos x)$$

$$v_2 = \int \frac{1 \cdot e^x \cos x \, dx}{2e^{2x}} = \frac{1}{2}\int e^{-x} \cos x \, dx = \frac{1}{2} \times \frac{1}{2} e^{-x}(\sin x - \cos x) = \frac{1}{4} e^{-x}(\sin x - \cos x)$$

$$\therefore y_p = -\frac{1}{4} e^x(\sin x + \cos x) + \frac{1}{4} e^x(\sin x - \cos x)$$

$$= -\frac{1}{4} e^x \cdot 2\cos x = -\frac{1}{2} e^x \cos x$$

$\therefore$ The required solution is

$$y = \text{C.F.} + y_p$$
$$= c_1 + c_2 e^{2x} - \frac{1}{2} e^x \cos x \quad \text{Ans}$$