CLASS NOTE

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

Numerical Integration

The analytical method of evaluating a definite integral $\int_a^b f(x)dx$ is to find a primitive $g(x)$ of $f(x)$ such that $g'(x)=f(x)$ and then Calculate $g(b)-g(a)$. But in many Cases it is found that the primitive of $f(x)$ can not be obtained in terms of known function. In some Cases, only the values of function at a set of points are given. In those Cases we have to use numerical integration.

The basic idea of numerical integration is to divide the interval $[a,b]$ into sub-intervals by inclusion of points $x_0, x_1, ----, x_n$ and then Calculate the values of the function $f(x)$ at these points, approximate $f(x)$ by interpolation polynomial $\varphi_n(x)$, then the given integral will be approximately $\int_a^b \varphi_n(x)\,dx$, which can be directly Calculated.

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

⊛ Deduction of Trapezoidal and Simpson's $1/3$rd rule from Newton's forward difference formula —

Suppose the interval $[a,b]$ be sub divided into n equal ~~sub~~ intervals by points $a=x_0, x_1, ----, x_{n-1}, x_n=b$, Where
$x_r = x_{r-1}+h$ for $r=1,2,---,n$
Then by Newton's forward difference formula we have,

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ------ \quad (1)$$

where $x_n = x_0 + nh$ and $x = x_0 + uh$ $---\rightarrow (2)$

Integrating both sides of (1) between x_0 to x_n we get,

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_n}\left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + ---\right]dx$$

$$= \int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \cdots\right] h\,du$$

$$= h\left[u y_0 + \frac{u^2}{2}\Delta y_0 + \frac{\frac{u^3}{3}-\frac{u^2}{2}}{2!}\Delta^2 y_0 + \frac{\frac{u^4}{4}-u^3+u^2}{3!}\Delta^3 y_0 + \cdots\right]_0^n$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{n^2(2n-3)}{12}\Delta^2 y_0 + \frac{n^2(n^2-4n+4)}{24}\Delta^3 y_0 + \cdots\right]$$

Now, $x = x_0 + uh$
$\therefore dx = h\,du$

x	x_0	x_n
u	0	n

[Using (2)]

P.T.O.

$$= nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right]$$

$$\therefore \int_{x_0}^{x_n} y\,dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots\right] \rightarrow (3)$$

From the above formula (3), we can derive different integration formula by putting $n = 1, 2, \ldots$ etc.

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

Derivation of Trapezoidal Rule :

putting $n=1$ in (3) we get,

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] = \frac{h}{2}(y_0 + y_1) \rightarrow (4)$$

Similarly, we can have,

$$\int_{x_1}^{x_2} y\,dx = \frac{h}{2}\left[y_1 + y_2\right] \rightarrow (5), \quad \int_{x_2}^{x_3} y\,dx = \frac{h}{2}\left[y_2 + y_3\right] \rightarrow (6)$$

$$\int_{x_{n-1}}^{x_n} y\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right] \rightarrow (7)$$

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

Combining all these integral we get,

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_1} y\,dx + \int_{x_1}^{x_2} y\,dx + \int_{x_2}^{x_3} y\,dx + \cdots + \int_{x_{n-1}}^{x_n} y\,dx$$

$$= \frac{h}{2}\left[y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots + y_{n-1} + y_n\right]$$

$$= \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})\right] \rightarrow (8)$$

This is known as Trapezoidal Rule.

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

Simpson's 1/3rd rule :

putting $n=2$ in (3) we get,

$$\int_{x_0}^{x_2} y\,dx = 2h\left[y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right] = \frac{h}{3}\left[6y_0 + 6(y_1 - y_0) + (y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \rightarrow (9)$$

Similarly,

$$\int_{x_2}^{x_4} y\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right] \rightarrow (10), \quad \int_{x_4}^{x_6} y\,dx = \frac{h}{3}\left[y_4 + 4y_5 + y_6\right] \rightarrow (11)$$

$$\int_{x_{n-2}}^{x_n} y\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right] \rightarrow (12)$$

P. T. O.

P. T. O.

$$\therefore \int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_2} y\,dx + \int_{x_2}^{x_4} y\,dx + \int_{x_4}^{x_6} y\,dx + \cdots + \int_{x_{n-2}}^{x_n} y\,dx$$

$$= \frac{h}{3}\Big[(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+(y_4+4y_5+y_6)+\cdots + (y_{n-2}+4y_{n-1}+y_n)\Big]$$

$$= \frac{h}{3}\Big[y_0+y_n+4(y_1+y_3+y_5+\cdots+y_{n-1})+2(y_2+y_4+y_6+\cdots+y_{n-2})\Big] \longrightarrow (13)$$

This is known as Simpson's $\frac{1}{3}$rd rule.

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

Error in Trapezoidal Rule:

$$E_T = \int_{x_0}^{x_n} y\,dx - \frac{h}{2}\left[y_0+y_n+2(y_1+y_2+\cdots+y_{n-1})\right] \longrightarrow (1)$$

Now, for the interval $[x_0, x_1]$, error is given by

$$E_1 = \int_{x_0}^{x_1} y\,dx - \frac{h}{2}(y_0+y_1) \longrightarrow (2)$$

Suppose $y=f(x)$ be continuous and possesses continuous derivatives in $[x_0,x_1]$, then expanding $y(x)$ about $x=x_0$ in Taylor series we get,

$$f(x_0+h) = f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\frac{h^3}{3!}f'''(x_0)+\cdots$$

$$E_1 = \int_{x_0}^{x_1}\left[y_0+(x-x_0)y_0'+\frac{(x-x_0)^2}{2!}y_0''+\frac{(x-x_0)^3}{3!}y_0'''+\cdots\right]dx - \frac{h}{2}(y_0+y_1)$$

$$= \left[xy_0+\frac{(x-x_0)^2}{2!}y_0'+\frac{(x-x_0)^3}{3!}y_0''+\frac{(x-x_0)^4}{4!}y_0'''+\cdots\right]_{x_0}^{x_1} - \frac{h}{2}(y_0+y_1)$$

$$= \left[(x_1-x_0)y_0+\frac{(x_1-x_0)^2}{2!}y_0'+\frac{(x_1-x_0)^3}{3!}y_0''+\frac{(x_1-x_0)^4}{4!}y_0'''+\cdots\right] - \frac{h}{2}(y_0+y_1)$$

$$= \left[hy_0+\frac{h^2}{2!}y_0'+\frac{h^3}{3!}y_0''+\frac{h^4}{4!}y_0'''+\cdots\right] - \frac{h}{2}(y_0+y_1)$$

$$= \left[hy_0+\frac{h^2}{2!}y_0'+\frac{h^3}{3!}y_0''+\frac{h^4}{4!}y_0'''+\cdots\right] - \frac{h}{2}\left[y_0+y(x_0+h)\right]$$

$$= \left[hy_0+\frac{h^2}{2!}y_0'+\frac{h^3}{3!}y_0''+\frac{h^4}{4!}y_0'''+\cdots\right] - \frac{h}{2}\left[y_0+y_0+hy_0'+\frac{h^2}{2!}y_0''+\frac{h^3}{3!}y_0'''+\cdots\right]$$

$$\overset{(*)}{=} \left[hy_0+\frac{h^2}{2!}y_0'+\frac{h^3}{3!}y_0''+\frac{h^4}{4!}y_0'''+\cdots\right] - \left[hy_0+\frac{h^2}{2}y_0'+\frac{h^3}{4}y_0''+\frac{h^4}{12}y_0'''+\cdots\right]$$

$$= \left(\frac{1}{6}-\frac{1}{4}\right)h^3y_0'' = -\frac{1}{12}h^3y_0''$$

This is the error in evaluating the integral in the interval (x_0, x_1).

(*) Neglecting higher order terms of $O(h^4)$ we get,

P.T.O.

Similarly, for the other intervals $(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n)$ we can compute the errors for Trapezoidal rule.

Thus total error in Trapezoidal rule is

$$E_T = \int_{x_0}^{x_n} y\,dx - \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 + \cdots + y_{n-1})\right]$$

$$= -\frac{h^3}{12}\left[y_0'' + y_1'' + y_2'' + \cdots + y_{n-1}''\right]$$

$$\leq -\frac{h^3}{12} n y''(\xi),$$ $\boxed{\text{where } y''(\xi) \text{ is the largest values of } n \text{ quantities } y_0'', y_1'', y_2'', \ldots, y_{n-1}''}$

$$\therefore E_T \simeq -\frac{nh^3}{12} y''(\xi), \quad a < \xi < b \text{ and } nh = b - a$$

$$= -\frac{(b-a)}{12} h^2 y''(\xi).$$

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agartala

Q. Find the error terms for Simpson's $\frac{1}{3}$rd rule.
[Hints: consult any book]

Q.1. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using (i) Trapezoidal rule, (ii) Simpson's $\frac{1}{3}$rd rule. Hence compute an approximate value of π in each case.

Sol^n: Here $f(x) = \frac{1}{1+x^2}$, $a = 0$, $b = 1$, $n = 6$ (say)

Then $h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$

The tabulated values of $f(x)$ for different values of x are given below:

x	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{5}{6}$	1
$f(x)$	1 (y_0)	0·972972 (y_1)	0·9 (y_2)	0·8 (y_3)	0·692308 (y_4)	0·590164 (y_5)	0·5 (y_6)

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agartala

(i) By Trapezoidal rule we have

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{12}\left[1 + 0{\cdot}5 + 2(0{\cdot}972972 + 0{\cdot}9 + 0{\cdot}8 + 0{\cdot}692308 + 0{\cdot}590164)\right]$$

$$= \frac{1}{12}\left[1{\cdot}5 + 7{\cdot}910888\right] = \frac{1}{12} \times 9{\cdot}410888 \simeq 0{\cdot}784241$$

Again, $\int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^1 = \tan^{-1}1 = \tan^{-1}\tan\frac{\pi}{4} = \frac{\pi}{4}$

$\therefore \frac{\pi}{4} \simeq 0{\cdot}784241$ ie $\pi \simeq 3{\cdot}136964$

P.T.O.

(ii) By Simpson's 1/3rd rule we have

$$\int_0^1 \frac{1}{1+x^2}\,dx = \frac{h}{3}\left[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{18}\left[1+0.5+4(0.972972+0.8+0.590164)+2(0.9+0.692302)\right]$$

$$= \frac{1}{18}\left[1.5+9.452544+3.184616\right] = \frac{1}{18}\times 14.13716$$

$$\simeq 0.785398$$

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

Again, $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$

$\therefore \frac{\pi}{4} \simeq 0.785398$ ie $\pi \simeq 3.141592$

— o —

Q.2. Evaluate $\int_0^{0.4} e^{x^3}\,dx$ to 5 decimal places by Trapezoidal rule taking $n=4$. Ans 0.40694

Q.3. Evaluate $\int_0^1 \frac{dx}{1+x}$ by Simpson's $\frac{1}{3}$rd rule with $h=0.25$. Also Calculate the absolute error. Ans 0.6931472, $E_A = 1.0672\times10^{-4}$.

Q.4. Evaluate $\int_0^{0.6} \frac{dx}{\sqrt{1-x^2}}$ using (i) Trapezoidal, (ii) Simpson's 1/3rd rule. Ans (i) 0.64374, (ii) 0.64350

Q.5. The following table gives the values of acceleration (f) of a particle in Cm/sec^2 at equal interval of time (t) in secs. Find the velocity of the particle at $t=2$ secs.

t	0.0	0.5	1.0	1.5	2.0
f	0.3989	0.3521	0.2420	0.1295	0.0540
	y_0	y_1	y_2	y_3	y_4

Soln: The velocity (v) at $t=2$ secs is given by

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

$$v = \int_0^2 f(t)\,dt$$

Since the number of sub-interval is 4, we use Simpson's 1/3rd rule.

$$\therefore \int_0^2 f(t)\,dt \simeq \frac{0.5}{3}\left[y_0+y_4+4(y_1+y_3)+2y_2\right]$$

$$= \frac{0.5}{3}\left[0.3989+0.0540+4(0.3521+0.1295)+2\times0.2420\right]$$

$$= 0.4772$$

$\therefore$ The required velocity is 0.4772 Cm/sec^2. Ans

— o —

Q.6. A Solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines $x=0$, $x=1$ and a curve through the points with following co-ordinates:

x	0	0·25	0·5	0·75	1
y	1	0·9896	0·9589	0·9089	0·8415

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agartala.

Find the volume of the solid of revolution.

Soln: The volume of the solid is $V=\pi\int_0^1 y^2\,dx$

Constructing table for y^2 (for given values of x) and using Simpson's $\frac{1}{3}$rd rule, $V=2·82038$.

Q.7. Compute the value of $\int_{1·2}^{1·6}\left(x+\frac{1}{x}\right)dx$, correct to five decimal places by using (i) Trapezoidal rule, (ii) Simpson's $\frac{1}{3}$rd rule.
Ans (i) 0·84794, (ii) 0·84768

Q.8. Find the value of $\int_0^{\pi/2}\sqrt{\sin x}\,dx$, taking $n=8$, correct to five significant figures, by using (i) Trapezoidal rule, (ii) Simpson's $\frac{1}{3}$rd rule. Ans (i) 1·1703, (ii) 1·1873

Q.9. Evaluate approximately by Trapezoidal rule, the integral $\int_0^1(4x-3x^2)dx$, by taking $n=10$. Compute the exact integral and hence find the absolute and relative error.
Ans 0·995, 1, 0·005, 0·005

Q.10. Using Simpson's $\frac{1}{3}$rd rule, estimate the area, bounded by the curve $y=f(x)$, x-axis and the ordinates $x=0$ and $x=2$ from the following data:

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,

x	0·0	0·5	1·0	1·5	2·0
$f(x)$	0·3989	0·3521	0·2420	0·3295	0·0540

Ans 0·4772.

— o —