Second order Differential Equation with Variable Coefficients

The general form of a second order differential equation is $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \dashrightarrow (1)$

where P, Q, R are functions of x only.

To solve (1) we assume $y = u(x)v(x)$

Then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

and $\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx}$

Substituting these values of y, $\frac{dy}{dx}$ & $\frac{d^2y}{dx^2}$ in (1) we get,

$$u\frac{d^2v}{dx^2} + v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + P\left(u\frac{dv}{dx} + v\frac{du}{dx}\right) + Quv = R$$

or, $$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R \longrightarrow (2)$$

Now, two cases may arise.

<u>Case-I</u> When $u(x)$ is known ie $u(x)$ is a known integral of the complementary function (C.F.) of the given equation (1).

Then we have $\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \longrightarrow (3)$

Thus, the equation (2) reduces to

$$u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + Pu\right)\frac{dv}{dx} = R$$

or, $$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\therefore \frac{dZ}{dx} + \left(\frac{2}{u}\frac{du}{dx} + P\right)Z = \frac{R}{u} \longrightarrow (4)$$

where $Z = \frac{dv}{dx}$, (Say)

Equation (4) is a first order linear equation, which can be solved easily for Z. Then by integrating $\frac{dv}{dx} = Z$, we can obtain v.

So the complete integral of (1) is $y = uv$.

P.T.O.

Case-II Here $u(x)$ is not known to us. We choose $u(x)$ in such a way that the coefficient of $\frac{dv}{dx}$ in ~~the~~ equation (2) vanishes. ie,

$$2\frac{du}{dx} + Pu = 0$$

or, $\frac{du}{u} + \frac{1}{2}P\,dx = 0$

Integrating we get,

$$\log u = -\frac{1}{2}\int P\,dx$$

$$\boxed{\therefore u = e^{-\frac{1}{2}\int P\,dx}}$$

Now, $\frac{du}{dx} = e^{-\frac{1}{2}\int P\,dx}\left(-\frac{1}{2}P\right) = -\frac{1}{2}Pu$

and $\frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\frac{du}{dx} + u\frac{dP}{dx}\right]$

$$= -\frac{1}{2}\left[P\left(-\frac{1}{2}Pu\right) + u\frac{dP}{dx}\right]$$

$$= \frac{1}{4}P^2u - \frac{1}{2}u\frac{dP}{dx}$$

Substituting these values in (2) we get,

$$u\frac{d^2v}{dx^2} + \left[\frac{1}{4}P^2u - \frac{1}{2}u\frac{dP}{dx} - \frac{1}{2}P^2u + Qu\right]v = R$$

or, $\frac{d^2v}{dx^2} + \left(Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}\right)v = \frac{R}{u}$

or, $\boxed{\frac{d^2v}{dx^2} + Lv = S} \to (3)$, where $\boxed{L = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}}$ $\boxed{S = \frac{R}{u} = Re^{\frac{1}{2}\int P\,dx}}$

Equation (3) is known as the normal form of the equation (1). When P, Q, R are known, then we can easily evaluate both u & v. Hence by putting those values of u & v in the relation $y = uv$, we can obtain the complete integral of equation (1).

Rules to determine $u(x)$:-

(i) If $1 + P + Q = 0$, then $u(x) = e^x$

(ii) If $1 - P + Q = 0$, then $u(x) = e^{-x}$

(iii) If $1 + \frac{P}{S} + \frac{Q}{S^2} = 0$, then $u(x) = e^{Sx}$, $S \neq 0$

(iv) If $P + Qx = 0$, then $u(x) = x$

(v) If $S(S-1) + PSx + Qx^2 = 0$, $u(x) = x^S$

1. Solve: $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 3y = 2\sec x$, $\to(1)$ given that $y = \sec x$ is a Solution. (ie solve in terms of known integral).

Solution: Here $P = -2\tan x$, $Q = 3$, $R = 2\sec x$

and $u = \sec x$

Let $y = v\sec x$ be the Complete integral of equation (1).

Then the equation (1) reduces to

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u}$$

or, $\frac{d^2v}{dx^2} + \left(\frac{2}{\sec x}\sec x\tan x - 2\tan x\right)\frac{dv}{dx} = \frac{2\sec x}{\sec x}$

or, $\frac{d^2v}{dx^2} = 2$

Integrating,

or, $\frac{dv}{dx} = 2x + c_1$

Integrating $v = x^2 + c_1x + c_2$

∴ The Complete integral of the given equation (1)

is $y = \sec x(x^2 + c_1x + c_2)$ Ans.

2. Solve: $\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{1}{x-1}y = x-1 \to (1)$

Soln:- Here $P = -\frac{x}{x-1}$, $Q = \frac{1}{x-1}$, $R = x-1$, we note that,

Here $u(x)$ is not given. But $P + Qx = 0$. So we take $u(x) = x$ as a Solution of the associated homogeneous equation of the given equation.

Let $y = vx$ be the Complete integral of (1).

Then (1) reduces to

$$\frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} = \frac{R}{u}$$

or, $\frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{x}{x-1}\right)\frac{dv}{dx} = \frac{x-1}{x}$

or, $\frac{dZ}{dx} + \left(\frac{2}{x} - \frac{x}{x-1}\right)Z = \frac{x-1}{x} \to (2)$

which is a linear equation, where $Z = \frac{dv}{dx}$

$$I.F. = e^{\int\left(\frac{2}{x} - \frac{x}{x-1}\right)dx}$$
$$= e^{2\log x - \int\left(1 + \frac{1}{x-1}\right)dx}$$
$$= e^{2\log x - x - \log(x-1)}$$
$$= e^{-x} e^{\log\frac{x^2}{x-1}}$$
$$= e^{-x}\frac{x^2}{x-1}$$

P.T.O.

From (2), $\frac{d}{dx}\left(z e^{-x}\frac{x^2}{x-1}\right) = xe^{-x}$ [By Multiplying both Sides of (2) by I.F.

Integrating we get,

$$z e^{-x}\frac{x^2}{x-1} = \int xe^{-x}dx + C_1$$
$$= -xe^{-x} - e^{-x} + C_1$$
$$= -(x+1)e^{-x} + C_1$$

or, $z = -\frac{(x^2-1)}{x^2} + C_1 e^{x}\left(\frac{x-1}{x^2}\right)$

or, $\frac{dv}{dx} = \frac{1}{x^2} - 1 + C_1 e^{x}\left(\frac{1}{x} - \frac{1}{x^2}\right)$

Integrating, $v = -\frac{1}{x} - x + C_1\frac{e^x}{x} + C_2$

∴ The required Complete integral of the given equation (1)

$$y = x\left[-\frac{1}{x} - x + C_1\frac{e^x}{x} + C_2\right]$$
$$= C_1 e^x + C_2 x - x^2 - 1$$ Ans

Solve the followings:

3. $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$, in terms of known integral.

4. $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 3y = 2\sec x$, given that $y = \sin x$ is a part of its Complementary function.

5. $x\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} - y = e^x$, interms of known integral.

6. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = e^{x^2}$ after reducing it to normal form.

7. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 4x^2y = xe^{x^2}$,, ,, ,, ,, ,,

8. $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2-1)y = -3e^{x^2}\sin 2x$,, ,, ,, ,, ,,

9. $\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (x^2+2)y = e^{\frac{x(x+2)}{2}}$,, ,, ,, ,, ,,

10. $\frac{d^2y}{dx^2} + 2\tan x\frac{dy}{dx} + (\tan^2 x + \sec^2 x)y = \sec x\tan x$, ,, ...