(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College. Agatala.

Numerical Solutions of Algebraic and Transcendental Equations

In engineering and applied science we frequently deal with the problem of finding one or more roots of the equation

$$f(x) = 0 \quad --- \rightarrow (1)$$

Here $f(x)$ is a non-linear function of real variable x, in general. But in most cases, it is difficult to find an explicit solutions of the equation (1) and therefore; we proceed to search for a root of (1) numerically with any specified degree of accuracy. The numerical methods of finding these roots are called iterative methods.

The function $f(x)$ may have any one of the following forms:

(i) $f(x)$ is an algebraic or polynomial function of degree n, say, so that $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1} + a_n x^n$ where a_i $(i=0,1,2,\ldots,n)$ are constants, real or complex and $a_n \neq 0$. For example, x^4+5x^3-9x+3, $x^{11}+15$ etc are algebraic functions. In such cases, the equation $f(x)=0$ is called algebraic equation.

(ii) $f(x)$ is a transcendental function, for example $\sin x + x^2+5$, $e^x + \log x + \cot x - x + 2$ etc. ~~are~~ In ~~these~~ these case, $f(x)=0$ is called transcendental equation i.e., $f(x)$ can be written as

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

where $a_0, a_1, a_2, \ldots$ are not constants but functions of x.

Some important results or theorems:

1. Every algebraic equation has at least one root.
2. An algebraic equation of degree n has exactly n roots.
3. If $\alpha + i\beta$ or $\alpha+\sqrt{\beta}$ $(\beta \neq 0, i=\sqrt{-1})$ be a root of the algebraic equation $f(x)=0$ with real coefficients then $\alpha - i\beta$ or $\alpha - \sqrt{\beta}$ will be another root of the equation.
4. If $f(a)$ and $f(b)$ are of opposite signs, then there are an odd number (at least one) of real roots of the equation $f(x)=0$ between a and b. If they have same sign then either there is no real root or there are an even number of real roots of the equation $f(x)=0$ between a and b.

P.T.O.

Bisection method: Consult any book.

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

Convergence of bisection method:

Suppose the interval $[a_n, b_n]$ contains the root α and $f(a_n)f(b_n) < 0$. Let $x_{n+1} = \frac{1}{2}(a_n + b_n)$. If $f(a_n)f(x_{n+1}) < 0$, then $a_{n+1} = a_n$, $b_{n+1} = x_{n+1}$. On the other hand, if $f(x_{n+1})f(b_n) < 0$, then $a_{n+1} = x_{n+1}$, $b_{n+1} = b_n$.

Thus in any case, $\alpha \in [a_{n+1}, b_{n+1}]$, $f(a_{n+1})f(b_{n+1}) < 0$ and

$$b_{n+1} - a_{n+1} = \frac{1}{2}(b_n - a_n) \quad \text{---(i)}$$

If $\alpha \in [a_n, x_{n+1}]$ then

$a_{n+1} = a_n$, $b_{n+1} = x_{n+1} = \frac{1}{2}(a_n + b_n)$

$\therefore b_{n+1} - a_{n+1} = \frac{1}{2}(a_n + b_n) - a_n = \frac{1}{2}(b_n - a_n)$

Again, if $\alpha \in [x_{n+1}, b_n]$ then

$a_{n+1} = x_{n+1} = \frac{1}{2}(a_n + b_n)$, $b_{n+1} = b_n$

$\therefore b_{n+1} - a_{n+1} = b_n - \frac{1}{2}(a_n + b_n) = \frac{1}{2}(b_n - a_n)$

Similarly
$$b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1})$$
$$b_{n-1} - a_{n-1} = \frac{1}{2}(b_{n-2} - a_{n-2})$$
$$\text{-------------}$$
$$b_1 - a_1 = \frac{1}{2}(b_0 - a_0)$$

From (i)
$$b_{n+1} - a_{n+1} = \frac{b_0 - a_0}{2^{n+1}}$$

Now, if $|h_n| = |x_{n+1} - x_n| = \frac{b_n - a_n}{2}$

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala.

$$|\varepsilon_n| = |\alpha - x_n| \leq b_n - a_n$$

$\therefore |\varepsilon_n| \leq \frac{b_0 - a_0}{2^n}$. So $\varepsilon_n \to 0$ as $n \to \infty$

Hence the iterative process converges and

$$|\varepsilon_{n+1}| \leq |h_n| = \frac{b_0 - a_0}{2^{n+1}} \to 0.$$

Note: $\frac{\varepsilon_{n+1}}{\varepsilon_n} = \frac{b_0 - a_0}{2^{n+1}} \times \frac{2^n}{b_0 - a_0} = \frac{1}{2}$, so the convergence in bisection method is linear.

Advantage and disadvantage of bisection method:

Advantage: This method is very simple, as at any stage of iteration the approximate value of the desired root of the equation $f(x) = 0$ does not depends on the values of $f(x_n)$ but on their signs only. Also the method is unconditionally and surely convergent.

Disadvantage: The method is very slow and requires large number of iteration to obtain moderately accurate results and hence it is laborious.

Q.1. Find a real root of the equation $x^3-4x-9=0$ correct to three decimal places by the method of bisection.

$\underline{\text{Sol}^n}$:- Let $f(x) = x^3-4x-9$. Then $f(0)=-9$, $f(1)=-12$, $f(2)=-9$, $f(3)=6$, $f(2.5)=-3.375$.

Since $f(2.5).f(3)<0$, so there lies a root between 2.5 and 3 of $f(x)=0$.

Let $a_0=2.5$, $b_0=3$ so that $x_1=\frac{1}{2}(2.5+3)=2.75$

No. of iteration (n)	a_n for which $f(a_n)<0$	b_n for which $f(b_n)>0$	$x_{n+1}=\frac{a_n+b_n}{2}$	$f(x_{n+1})$
0	2.5	3	2.75	0.7969
1	2.5	2.75	2.625	−1.4121
2	2.625	2.75	2.6875	−0.3391
3	2.6875	2.75	2.71875	0.2209
4	2.6875	2.71875	2.703125	−0.0610
5	2.703125	2.71875	2.7109	0.0787
6	2.703125	2.7109	2.7070	0.0084
7	2.703125	2.7070	2.7051	−0.0256
8	2.7051	2.707	2.7061	−0.0077
9	2.7061	2.707	2.7065	−0.0005
10	2.7065	2.707	2.70675	0.0039
11	2.7065	2.70675	2.7066	

Thus a real root of the given equation is 2.706 correct to three decimal places.

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.B. College, Agatala,

Q.2. Find a real root of the following equations using bisection method:

(i) $f(x)=x^3-x-1=0$, (ii) $x^3-2x-5=0$

(iii) $x^3-3x-5=0$, (iv) $x^3-9x^2-46x+120=0$

(JAYANTA SAHA)
Assistant Professor,
Dept. of Mathematics,
M.B.R. College, Agatala.