Change of independent variable

We consider the second order linear equation with variable coefficients ie

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \quad \dashrightarrow (1)$$

Let $z = f(x)$

$$\frac{dz}{dx} = f'(x), \quad \frac{d^2z}{dx^2} = f''(x)$$

Now, $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = f'(x)\frac{dy}{dz}$

$$\frac{d^2y}{dx^2} = f''(x)\frac{dy}{dz} + f'(x)\frac{d^2y}{dz^2}\frac{dz}{dx}$$

$$= f''(x)\frac{dy}{dz} + \{f'(x)\}^2\frac{d^2y}{dz^2}$$

Now, putting these values in equation (1) we get,

$$f''(x)\frac{dy}{dz} + \{f'(x)\}^2\frac{d^2y}{dz^2} + Pf'(x)\frac{dy}{dz} + Qy = R$$

or, $\{f'(x)\}^2\frac{d^2y}{dz^2} + \{f''(x) + Pf'(x)\}\frac{dy}{dz} + Qy = R$

or, $\frac{d^2y}{dz^2} + \frac{f''(x) + Pf'(x)}{\{f'(x)\}^2}\frac{dy}{dz} + \frac{Q}{\{f'(x)\}^2}y = \frac{R}{\{f'(x)\}^2}$

or, $\frac{d^2y}{dz^2} + \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$

$$\therefore \frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad \dashrightarrow (2)$$

where $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$, $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Now, if we choose z in such a way that the coefficient of $\frac{dy}{dz}$ in the transformed equation(2) vanishes then $P_1 = 0$ and eqn (2) become $\frac{d^2y}{dz^2} + Q_1 y = R_1 \dashrightarrow (3)$

ie $\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0 \dashrightarrow (4)$

P.T.O.

From (4)

or, $\frac{dL}{dx} + PL = 0$, where $L = \frac{dz}{dx}$, (say)

or, $\frac{dL}{L} = -P\,dx$

Integrating $\log L = -\int P\,dx + \log A$

or, $L = A e^{-\int P dx}$

Taking $A = 1$, $L = e^{-\int P dx}$ i.e, $\frac{dz}{dx} = e^{-\int P dx}$ ---> (A)

i.e $z = \int e^{-\int P dx}\,dx$

Equation (3) can be solved if $Q_1 = K$ (Constant) or $Q_1 = \frac{K}{z^2}$ (i.e transformed equation will be linear homogeneous equation).

Further, if P_1 & Q_1 become Constant for any choice of z, then the transformed equation can be solved easily. In this case the transformed equation will be a linear equation with constant coefficients.

1. Solve: $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0$ by changing the independent variable.

Solution: Given $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y\cos^2 x = 0$ ---> (1)

Here $P = \tan x$, $Q = \cos^2 x$, $R = 0$

Let $z = f(x)$ be such that $\frac{dz}{dx} = e^{-\int P dx}$

$$= e^{-\int \tan x\, dx}$$

$$= e^{-\log \sec x} = \cos x$$

$\therefore z = \sin x$

Then the transformed equ^n will be

$\frac{d^2y}{dz^2} + Q_1 y = R_1$, where $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos^2 x}{(\cos x)^2} = 1$

or, $\frac{d^2y}{dz^2} + y = 0$ and $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$

$\therefore y = C_1 \cos z + C_2 \sin z$

$= C_1 \cos(\sin x) + C_2 \sin(\sin x)$ which is the required Sol^n.

2. Solve: $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0$ by changing the independent variable.

Solution: $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0 \quad \cdots\rightarrow(1)$

Here $P = \cot x$, $Q = 4\operatorname{cosec}^2 x$, $R = 0$

Let $z = f(x)$ so that $\frac{dz}{dx} = e^{-\int P dz} = e^{-\int \cot x\, dx} = e^{-\log \sin x}$

$= \operatorname{cosec} x$

$\therefore z = \int \operatorname{cosec} x\, dx = \log(\operatorname{cosec} x - \cot x)$

$= \log \tan x/2$

$\therefore$ The equ^n (1) reduces to

$\frac{d^2y}{dz^2} + Q_1 y = R_1$

or, $\frac{d^2y}{dz^2} + 4y = 0$

or, $y = c_1 \cos 2z + c_2 \sin 2z$

$\therefore y = c_1 \cos\left(2\log \tan \frac{x}{2}\right) + c_2 \sin\left(2\log \tan \frac{x}{2}\right)$,

which is the required solution.

where $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2}$

$= \frac{4\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x} = 4$

$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$

3. Solve: $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2 y = x^4 \quad --\rightarrow(1)$

Solution: Here $P = -\frac{1}{x}$, $Q = 4x^2$, $R = x^4$

Let $z = f(x)$ so that that $\frac{dz}{dx} = e^{-\int P dx} = e^{-\int -\frac{1}{x}dx} = e^{\log x} = x$

$\therefore z = \frac{x^2}{2}$

Then equ^n (1) reduces to

$\frac{d^2y}{dz^2} + Q_1 y = R_1$

or, $\frac{d^2y}{dz^2} + 4y = 2z$

where $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{x^2} = 4$

$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{x^2} = x^2 = 2z$

$\therefore y = c_1 \cos 2z + c_2 \sin 2z + \frac{1}{D^2+4} 2z, \quad D \equiv \frac{d}{dz}$

$= c_1 \cos 2z + c_2 \sin 2z + \frac{2}{4}\left(1 - \frac{D^2}{4} + \frac{D^4}{16} - \cdots \infty\right) z$

$= c_1 \cos 2z + c_2 \sin 2z + \frac{1}{2} z$

$= c_1 \cos(x^2) + c_2 \sin(x^2) + \frac{1}{4}x^2$, which is the required solution.

4. Solve: $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\{\log(1+x)\}$ by changing the independent variable.

Sol^n:- Given $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\{\log(1+x)\}$

$$\text{or, } \frac{d^2y}{dx^2}+\frac{1}{(1+x)}\frac{dy}{dx}+\frac{1}{(1+x)^2}y=\frac{4\cos\{\log(1+x)\}}{(1+x)^2} \dashrightarrow (1)$$

Here $P=\frac{1}{1+x}$, $Q=\frac{1}{(1+x)^2}$, $R=\frac{4\cos\{\log(1+x)\}}{(1+x)^2}$

Let $z=f(x)$ be such that $\frac{dz}{dx}=e^{-\int P\,dx}=e^{-\int\frac{dx}{1+x}}$

$$=e^{-\log(1+x)}=\frac{1}{1+x}$$

$$\therefore z=\log(1+x)\;[=f(x)]$$

Thus the equ^n (1) reduces to

$$\frac{d^2y}{dz^2}+Q_1y=R_1$$

where $Q_1=\frac{Q}{\left(\frac{dz}{dx}\right)^2}=\frac{\frac{1}{(1+x)^2}}{\frac{1}{(1+x)^2}}=1$

$$R_1=\frac{R}{\left(\frac{dz}{dx}\right)^2}=\frac{\frac{4\cos\{\log(1+x)\}}{(1+x)^2}}{\frac{1}{(1+x)^2}}=4\cos\{\log(1+x)\}=4\cos z$$

$$\text{or, } \frac{d^2y}{dz^2}+y=4\cos z$$

$$\therefore y=C_1\cos z+C_2\sin z+\frac{1}{\theta^2+1}4\cos z$$

$$\theta\equiv\frac{d}{dz}$$

$$=C_1\cos z+C_2\sin z+4z\frac{1}{2\theta}\cos z$$

$$=C_1\cos z+C_2\sin z+2z\sin z$$

$$=C_1\cos\{\log(1+x)\}+C_2\sin\{\log(1+x)\}+2\log(1+x)\sin\{\log(1+x)\}.$$

which is the required solution.

5. Solve: $\frac{d^2y}{dx^2}-(1+4e^x)\frac{dy}{dx}+3e^{2x}y=e^{2(x+e^x)}$ by changing the independent variable.

Sol^n:- Given $\frac{d^2y}{dx^2}-(1+4e^x)\frac{dy}{dx}+3e^{2x}y=e^{2(x+e^x)} \dashrightarrow (1)$

Here $P=-(1+4e^x)$, $Q=3e^{2x}$, $R=e^{2(x+e^x)}$

Let us choose $\frac{dz}{dx}$ in such a way that the coefficient of y in the transformed equation become constant.

P.T.O

Let $\frac{dz}{dx} = e^x$ so that $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{3e^{2x}}{e^{2x}} = 3$

$\therefore z = e^x$

$$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{e^x - (1+4e^x)e^x}{e^{2x}}$$

$$= \frac{e^x(1-1-4e^x)}{e^{2x}} = -\frac{4e^{2x}}{e^{2x}}$$

$$= -4$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{e^{2(x+e^x)}}{e^{2x}} = e^{2e^x}$$

$\therefore$ The transformed equⁿ is obtain from given equⁿ by putting $z = e^x$ is

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

or, $\frac{d^2y}{dz^2} - 4\frac{dy}{dz} + 3y = e^{2e^x} = e^{2z}$

A.E. is $m^2 - 4m + 3 = 0$ i.e. $m = 1, 3$

$$\therefore y = c_1 e^z + c_2 e^{3z} + \frac{1}{\theta^2 - 4\theta + 3} e^{2z}, \quad \theta \equiv \frac{d}{dz}$$

$$= c_1 e^z + c_2 e^{3z} + \frac{1}{(\theta-1)(\theta-3)} e^{2z}$$

$$= c_1 e^z + c_2 e^{3z} + \frac{1}{(2-1)(2-3)} e^{2z}$$

$$= c_1 e^z + c_2 e^{3z} - e^{2z}$$

$$= c_1 e^{e^x} + c_2 e^{3e^x} - e^{2e^x}$$

Which is the required Complete integral. Ans

6. Solve: $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4}y = 0 \rightarrow (1)$ by changing independent variables.

Solution:- Here $P = \frac{2}{x}$, $Q = \frac{a^2}{x^4}$, $R = 0$

Let $\frac{dz}{dx} = e^{-\int P dx} = e^{-\int \frac{2dx}{x}} = e^{-2\log x} = e^{\log \frac{1}{x^2}} = \frac{1}{x^2}$

$\therefore z = -\frac{1}{x}$

Then the given equation reduces to

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

or, $\frac{d^2y}{dz^2} + a^2 y = 0$

where $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{a^2}{x^4}}{\frac{1}{x^4}} = a^2$

and $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$

$\therefore y = C_1 \cos az + C_2 \sin az$

$= C_1 \cos a(-\frac{1}{x}) + C_2 \sin a(-\frac{1}{x})$

$= C_1 \cos(\frac{a}{x}) - C_2 \sin(\frac{a}{x})$

Which is the required Complete integral.

Q.7. Solve: $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2 \sin x^2$ by changing the independent variable.

Solution: Given $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2 \sin x^2$ ---> (1)

Here $P = -\frac{1}{x}$, $Q = -4x^2$, $R = 8x^2 \sin x^2$

We choose $\frac{dz}{dx} = x$ so that in the transformed eqn the coefficient y become constant.

$\therefore z = \frac{x^2}{2}$

So the transformed equation will be

$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$

where

$P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{(\frac{dz}{dx})^2}$

$= \frac{1 - \frac{1}{x} \cdot x}{x^2}$

$= 0$

$Q_1 = \frac{Q}{(\frac{dz}{dx})^2}$

$= \frac{-4x^2}{x^2} = -4$

$R_1 = \frac{R}{(\frac{dz}{dx})^2}$

$= \frac{8x^2 \sin x^2}{x^2}$

$= 8 \sin(x^2)$

$= 8 \sin 2z$

or, $\frac{d^2y}{dz^2} - 4y = 8 \sin 2z$

A.E. is $m^2 - 4 = 0$ or, $m = \pm 2$

$\therefore y = C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{D^2 - 4} 8 \sin 2z$

$= C_1 e^{2z} + C_2 e^{-2z} + 8 \cdot \frac{1}{-4-4} \sin 2z$

$= C_1 e^{2z} + C_2 e^{-2z} - \sin 2z$

$= C_1 e^{x^2} + C_2 e^{-x^2} - \sin(x^2)$

Which is the required Complete integral.

Scanned with CamScanner