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The method of fixed point iteration or successive approximation

To find the roots of the equation $f(x)=0$, $\to(1)$ we rewrite the given equation in the form

$$x=\varphi(x) \longrightarrow (2)$$

Now, if α be a root of equation (1) then $\alpha=\varphi(\alpha)$ (ie α is a fixed point under the mapping φ)

Let $[a_0,b_0]$ be the initial interval containing the root α and $\varphi(x)$ is continuously differentiable for sufficient number of times in $[a_0,b_0]$ such that $x\in[a_0,b_0]$, $\varphi(x)\in[a_0,b_0]$ and $\varphi'(x)\neq 0$ in the interval.

Let $x=x_0$ $(a_0\leq x_0\leq b_0)$ be the initial approximation to the root α. put $x=x_0$ $(=a_0$ or $b_0)$ on the R.H.S. of (2) and get the first approximation as $x_1=\varphi(x_0)$. Thus the successive approximations are

$$x_2=\varphi(x_1),\ x_3=\varphi(x_2),\ \dots,\ x_{n+1}=\varphi(x_n)$$

ie. the iteration formula is $x_{n+1}=\varphi(x_n) \longrightarrow (3)$

Here x_n is the nth approximation of the root α of $f(x)=0$. But the sequence $\{x_n\}$ may or may not converge. If $\{x_n\}$ converges, then it must converge to α so that in the limit $\alpha=\varphi(\alpha)$.

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Convergence of fixed point iteration:

Theorem: Let $x=\alpha$ be a root of the equation $f(x)=0$ which is rewritten as $x=\varphi(x)$. If $\varphi(x)$ is continuous and $|\varphi'(x)|\leq l$, where $0<l<1$, in an interval I containing α, then the sequence $\{x_n\}$ of iterations determined from $x_{n+1}=\varphi(x_n)$, $(n=0,1,2,\dots)$ converges to the root α. (This condition of convergence of the sequence $\{x_n\}$ is sufficient only.)

Proof: Since α is a root of the equation $x=\varphi(x)$, ie $\alpha=\varphi(\alpha)$.

Now, $x_{n+1}-\alpha=\varphi(x_n)-\varphi(\alpha)$.

$=(x_n-\alpha)\varphi'(\xi)$, $x_n<\xi<\alpha$ (By Lagrange's MVT)

or, $|x_{n+1}-\alpha|=|x_n-\alpha||\varphi'(\xi)|\leq l|x_n-\alpha|$ $(\because |\varphi'(x)|\leq l)$

Thus $|x_n-\alpha|\leq l|x_{n-1}-\alpha|$, $|x_{n-1}-\alpha|\leq l|x_{n-2}-\alpha|$, $\dots$ $|x_1-\alpha|\leq l|x_0-\alpha|$

P.T.O.

So that $|x_{n+1}-\alpha| \le l^{n+1}|x_0-\alpha| \to 0$ as $n\to\infty$ $(\because 0<l<1)$

ie $x_{n+1}\to\alpha$ as $n\to\infty$.

Hence the sequence $\{x_n\}$ of iteration converges to the root α if $0<l<1$.

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Note :- 1) The sequence $\{x_n\}$ of iterations is rapidly convergent if $|\phi'(x)|$ is close to zero and slow if $|\phi'(x)|$ is close to 1.

2) If x_0 is very close to α then the number of iterations required for the convergence will be minimum.

3) The method of fixed point iteration is conditionally convergent and the condition of convergence is $|\phi'(x)|<1$ in the neighbourhood of α.

Error in fixed point iteration method:

Let x_{n+1} be the $(n+1)$th approximation of the root α of the equation $f(x)=0$ ie of $x=\phi(x)$. Then the corresponding error is given by

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$$\varepsilon_{n+1} = \alpha - x_{n+1}$$

or, $|\varepsilon_{n+1}| = |\alpha - x_{n+1}| \le l|\alpha - x_n|$, provided the iteration converges.

Now, $|\varepsilon_{n+1}| \le l|\alpha - x_{n+1} + x_{n+1} - x_n|$ $[\because |a+b|\le|a|+|b|]$

$\le l\{|\alpha - x_{n+1}| + |x_{n+1}-x_n|\}$

$\therefore$ ~~$|\varepsilon_{n+1}| \le l|\alpha - x_{n+1}| + l|x_{n+1}-x_n|$~~

or, $|\varepsilon_{n+1}| \le l\{|\varepsilon_{n+1}| + |h_n|\}$, putting $h_n = x_{n+1}-x_n$

or, $(1-l)|\varepsilon_{n+1}| \le l|h_n|$

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$\therefore$ $|\varepsilon_{n+1}| \le \dfrac{l}{1-l}|h_n|$

which is the error in approximating α by x_{n+1}.

Advantage:- The method is rapidly convergent if the initial approximation x_0 is very close to the desired root. Also the method is self-corrector, ie if at any stage there is an computational error in x_n, then the error is corrected in the next stage.

Disadvantage: The method is conditionally convergent. ie, it will converge if $|\phi'(x)|<1$. But some times it is difficult to express the equation $f(x)=0$, in the form $x=\phi(x)$.

Problem 1. Find the positive root of the equation $x^3+x-1=0$ by fixed point iteration method correct to three decimal places.

Solution :- Let $f(x) = x^3+x-1$, $f(0)=-1$, $f(1)=1$, $f(0.5)=-0.375$.

Thus a positive root lies between 0.5 and 1.

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Now, from the given equation

$$x(x^2+1) = 1$$

$$\text{or, } x = \frac{1}{x^2+1} = \varphi(x), \text{ (Say)}$$

$$\varphi'(x) = -\frac{2x}{(x^2+1)^2} \text{ and } |\varphi'(x)| < 1 \text{ in } (0.5, 1)$$

So we can apply fixed point iteration method.

Let $x_0 = 0.5$. Then $x_1 = \dfrac{1}{(0.5)^2+1} = 0.8$

$$x_2 = \frac{1}{(0.8)^2+1} = 0.60976, \quad x_3 = \frac{1}{(0.60976)^2+1} = 0.72896$$

$$x_4 = \frac{1}{(0.72896)^2+1} = 0.65300, \quad x_5 = \frac{1}{(0.653)^2+1} = 0.70106$$

$$x_6 = \frac{1}{(0.70106)^2+1} = 0.67047, \quad x_7 = \frac{1}{(0.67047)^2+1} = 0.68988$$

$$x_8 = \frac{1}{(0.68988)^2+1} = 0.67754, \quad x_9 = \frac{1}{(0.67754)^2+1} = 0.68537$$

$$x_{10} = \frac{1}{(0.68537)^2+1} = 0.68039, \quad x_{11} = \frac{1}{(0.68039)^2+1} = 0.68356$$

$$x_{12} = \frac{1}{(0.68356)^2+1} = 0.68155, \quad x_{13} = \frac{1}{(0.68155)^2+1} = 0.68282$$

$$x_{14} = \frac{1}{(0.68282)^2+1} = 0.68201, \quad x_{15} = \frac{1}{(0.68201)^2+1} = 0.68253$$

$$x_{16} = \frac{1}{(0.68253)^2+1} = 0.68220$$

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$\therefore$ The required root of the given equation is 0.682 Correct to three decimal places.

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Problem.2. Find a real root of the equation $\cos x = 2x-3$ Correct upto three decimal places using iteration method.

Solution :- Here $f(x) = \cos x - 2x + 3$

$f(0) = 4$, $f(1) = 1.5403$, $f(2) = -1.4161$ ..., $f(1.5) = 0.0707$

Hence a real root lies between 1.5 and 2.

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Given equation can be expressed as

$$x = \frac{1}{2}(\cos x + 3) = \phi(x), \text{ (Say)}$$

Now, $\phi'(x) = -\frac{1}{2}\sin x$ and $|\phi'(x)| = |-\frac{1}{2}\sin x| \leq \frac{1}{2} < 1$

Hence the iterative method is applicable.

So choose $x_0 = 1.5$ and using the formula

$x_{n+1} = \phi(x_n)$ we have

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$$x_1 = \frac{1}{2}(\cos 1.5 + 3) = \frac{0.07074 + 3}{2} = 1.53537$$

$$x_2 = \frac{1}{2}(\cos 1.53537 + 3) = \frac{1}{2}(0.03542 + 3) = 1.51771$$

$$x_3 = \frac{1}{2}(\cos 1.51771 + 3) = \frac{1}{2}(0.05306 + 3) = 1.52653$$

$$x_4 = \frac{1}{2}(\cos 1.52653 + 3) = \frac{1}{2}(0.04425 + 3) = 1.52212$$

$$x_5 = \frac{1}{2}(\cos 1.52212 + 3) = \frac{1}{2}(0.04866 + 3) = 1.52433$$

$$x_6 = \frac{1}{2}(\cos 1.52433 + 3) = \frac{1}{2}(0.04645 + 3) = 1.52322$$

$$x_7 = \frac{1}{2}(\cos 1.52322 + 3) = \frac{1}{2}(0.04756 + 3) = 1.52378$$

$$x_8 = \frac{1}{2}(\cos 1.52378 + 3) = \frac{1}{2}(0.04699 + 3) = 1.52349$$

$$x_9 = \frac{1}{2}(\cos 1.52349 + 3) = \frac{1}{2}(0.04729 + 3) = 1.52364$$

$$x_{10} = \frac{1}{2}(\cos 1.52364 + 3) = \frac{1}{2}(0.04714 + 3) = 1.52357$$

$\therefore$ The required root of the given equation is 1.524 Correct upto three decimal places.