Solve the following equations (ie find a real root) using fixed point iteration method correct to four decimal places:

3. $2x - \log_{10} x = 7$ in $[3,4]$, Ans 3·7892
4. $x^3 + 3x - 5 = 0$ in $[1,2]$ Ans 1·1538
5. $9x^2 - 3x - 2 = 0$ starting from $x_0 = 0$
6. $\sin x = 10(x-1)$ in $[1,2]$ Ans 1·0881
7. $\log x = \cos x$ in $[1,2]$ Ans 1·3024

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Newton-Raphson method :-

Let x_0 be the initial approximation of the desired root α of the given equation $f(x)=0$ and the correct root is $x_1 = x_0 + h$ so that

$$f(x_1) = f(x_0+h) = 0$$

By Taylor's series expansion we have

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots$$

$\min(x_1,x_0) < h < \max(x_1,x_0)$

$$\therefore f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots = 0,$$

[where $\min(x_1,x_0) < h < \max(x_1,x_0)$]

Neglecting the second and higher order terms, we have

$$f(x_0) + hf'(x_0) = 0 \text{ ie } h = -\frac{f(x_0)}{f'(x_0)}$$

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Thus $x_1 = x_0 + h = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$$\boxed{\therefore x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$ (ie 1st approximation)

Repeating the above process we obtain

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \text{ (ie, 2nd approximation.)}$$

Successive approximations are given by $x_3, x_4, \ldots, x_{n+1}$

where $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (ie, $(n+1)$th approximation) $\longrightarrow$ (A)

The sequence $\{x_n\}$ of successive corrections of the approximate root x_0 provides the correct root if (the sequence $\{x_n\}$) it is convergent. The result (A) is known as Newton-Raphson iteration formula.

Note:-

1) If $[a_0, b_0]$ be the initial interval in which the root α of the given equation $f(x)=0$ lies and $f'(x) \neq 0$, then the initial approximation may be started with $x_0 = a_0$ or, b_0.

Convergence of Newton-Raphson method :

We have $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \rightarrow (1)$ (from Newton-Raphson method formula)

and $x_{n+1} = \varphi(x_n) \rightarrow (2)$ (which is true for iterative method)

$\therefore \varphi(x_n) = x_n - \dfrac{f(x_n)}{f'(x_n)}$ (From above two relation)

ie $\varphi(x) = x - \dfrac{f(x)}{f'(x)}$

Now, $$\varphi'(x) = 1 - \frac{f'(x) \cdot f'(x) - f(x) f''(x)}{[f'(x)]^2}$$

$$= \frac{[f'(x)]^2 - [f'(x)]^2 + f(x) f''(x)}{[f'(x)]^2}$$

$$= \frac{f(x) f''(x)}{[f'(x)]^2}$$

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Now, ~~the~~ $\{x_n\}$ of Newton-Raphson method converges if and only if $|\varphi'(x)| < 1$ ie $|f(x)|.|f''(x)| < |f'(x)|^2$ in an interval in which the root lies.

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Error Estimation for Newton-Raphson method :

Let α be a root of the given equation $f(x)=0$. Then $f(\alpha)=0$.

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Now, $f(\alpha) = 0$

or, $f(x_n + \alpha - x_n) = 0$

or, $f(x_n) + (\alpha - x_n) f'(x_n) + \dfrac{(\alpha - x_n)^2}{2!} f''(\xi_n) = 0$, $[\min(\alpha, x_n) < \xi_n < \max(\alpha, x_n)]$ [Using Taylor's theorem]

or, $-\dfrac{f(x_n)}{f'(x_n)} = (\alpha - x_n) + \dfrac{1}{2}(\alpha - x_n)^2 \dfrac{f''(\xi_n)}{f'(x_n)} \rightarrow (1)$

P.T.O.

Again, from Newton-Raphson iteration formula we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow (2)$$

$$= \cancel{x_n} + \alpha - \cancel{x_n} + \frac{1}{2}(\alpha - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)}, \quad [\text{using } (1) \text{ in } (2)]$$

$$\therefore x_{n+1} - \alpha = \frac{1}{2}(\alpha - x_n)^2 \frac{f''(\xi_n)}{f'(x_n)} \longrightarrow (3)$$

Let $\alpha - x_n = \varepsilon_n$ and so $\alpha - x_{n+1} = \varepsilon_{n+1}$

From (3), $-\varepsilon_{n+1} = \frac{1}{2} \varepsilon_n^2 \frac{f''(\xi_n)}{f'(x_n)}$

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$$\therefore \varepsilon_{n+1} = -\frac{1}{2} \varepsilon_n^2 \frac{f''(\xi_n)}{f'(x_n)}$$

This is the error in Newton-Raphson formula.

If the iteration converges then $x_n, \xi_n \to \alpha$ as $n \to \infty$ So that

$$\lim_{n\to\infty} \left| \frac{\varepsilon_{n+1}}{\varepsilon_n^2} \right| = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$$

Hence Newton-Raphson iteration method is a second-order iteration process, ie. the convergence is quadratic and the constant asymptotic error is equal to $\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$.

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Note:-

1) NR method fails when $f'(x)=0$ or nearly equal to zero in the neighbourhood of the real root because in this case the connection h would be large and therefore, the iteration process is slow. Hence the method should not be used when the graph of $y=f(x)$ is nearly horizontal.

2) When $f'(x)$ is large in the neighbourhood (nbd) of the real root, ie, when the graph of the function $y=f(x)$ is nearly vertical, then it Crosses the x-axis. In this case, the method is very useful and the correct value of the root can be obtained more rapidly.

3) Newton-Raphson (NR) iterative sequence $\{x_n\}$ converges provided that the initial approximation x_0 is choosen sufficiently close to the root. otherwise, the sequence may diverge.

Advantage and disadvantage of NR method:

Advantage: Since the rate of convergence of this method is quadratic, the method converges more rapidly.

Disadvantage: In NR method, the initial approximation must be chosen very close to the root; otherwise, the method will fail. Since the method depends on the derivative $f'(x)$, it may not be suitable for a function $f(x)$ whose derivative is difficult to compute. Also the method fails if $f'(x)=0$ or small in the neighbourhood of the root.

Problem 1. Use NR method to find a positive root of the equation $e^x - 3x = 0$ correct to four decimal places.

Solution: Let $f(x) = e^x - 3x$, $f'(x) = e^x - 3$

$f(0) = 1$, $f(1) = -0.2817$, $f(0.5) = 0.1487$, $f(2) = 1.38906$

$\because f(0).f(1) < 0$, and $f(1).f(2) < 0$. So a positive root lies between $[0, 1]$ or, $[1, 2]$.

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We consider $[1, 2]$.

$f(1.5) = -0.01831$. So $f(1.5).f(2) < 0$ and hence a positive root lies between 1.5 and 2.

Take $x_0 = 1.5$

$$x_1 = x_0 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{-0.01831}{1.48169}$$

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$$= 1.5 + 0.01236 = 1.51236$$

$$x_2 = 1.51236 - \frac{f(1.51236)}{f'(1.51236)} = 1.51236 - \frac{0.00035}{1.53743}$$

$$= 1.51236 - 0.00023 = 1.51213$$

$$x_3 = 1.51213 - \frac{f(1.51213)}{f'(1.51213)} = 1.51213 - \frac{-0.000007}{1.53638}$$

$$= 1.51213 - 0.000004 = 1.512126$$

$\therefore$ The required root of the given equation is 1.5121 correct upto four decimal places.

Problem 2. Find the cube root of 10 upto 5 significant figures by NR method.

Solution: Let $x = \sqrt[3]{10}$ So that $x^3 = 10$ ie $x^3 - 10 = 0$

Let $f(x) = x^3 - 10 = 0$, $f'(x) = 3x^2$

$f(0) = -10$, $f(1) = -9$, $f(2) = -2$, $f(3) = 17$, $f(2.5) = 5.625$

Hence a real positive root of the equation $f(x) = 0$ lies between 2 and 2.5. ($\because f(2).f(2.5) < 0$)

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Take $x_0 = 2$

So that $x_1 = x_0 - \dfrac{f(2)}{f'(2)} = 2 - \dfrac{(-2)}{12} = 2 + 0.166667$
$= 2.166667$

$x_2 = 2.166667 - \dfrac{f(2.166667)}{f'(2.166667)} = 2.166667 - \dfrac{0.171301}{14.083337}$

$= 2.166667 - 0.012163 = 2.154504$

$x_3 = 2.154504 - \dfrac{f(2.154504)}{f'(2.154504)} = 2.154504 - \dfrac{0.000965}{13.925662}$

$= 2.154504 - 0.0000069 = 2.154497$

$x_4 = 2.154497 - \dfrac{f(2.154497)}{f'(2.154497)} = 2.154497 - \dfrac{0.000868}{13.925572}$

$= 2.154497 - 0.0000623 = 2.154435$

$x_5 = 2.154435 - \dfrac{f(2.154435)}{f'(2.154435)} = 2.154435 - \dfrac{0.0000043}{13.924771}$

$= 2.154435 - 0.0000003 = 2.154435$

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$\therefore x = \sqrt[3]{10} \simeq 2.15443$ Correct to five significant figures.

Problem 3. Using NR-method, find a real root of the equation $x^4 - x - 1 = 0$ which lies between 1 and 1.5 Correct to seven significant figures. Ans 1.220744

Problem 4. Using NR method, obtain iterative formula for the reciprocal of a number N and hence find the value of $\frac{1}{22}$, Correct to seven significant figures.

Solution: Let $\frac{1}{N} = x$ ie $\frac{1}{x} = N$ ie $\frac{1}{x} - N = 0$

Let $f(x) = \frac{1}{x} - N = 0$, $f'(x) = -\frac{1}{x^2}$

So by NR iterative formula we have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{\frac{1}{x_n} - N}{-\frac{1}{x_n^2}}$$

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$$= x_n + x_n^2\left(\frac{1}{x_n} - N\right)$$

$$= x_n + x_n - Nx_n^2 = (2 - Nx_n)x_n, \quad n = 0, 1, 2, \cdots$$

$\therefore x_{n+1} = (2 - Nx_n)x_n$, $n = 0, 1, 2, \cdots$

Which is the required iterative formula.

Second part:

Since $\frac{1}{22}$ lies between $\frac{1}{25}$ and $\frac{1}{20}$ ie between 0·04 and 0·05, So we take $x_0 = 0{\cdot}04$

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$\therefore \cancel{x_1 = 0{\cdot}04 - \frac{f(0{\cdot}04)}{f'(0{\cdot}04)} = 0{\cdot}04 -}$

$\therefore x_1 = (2 - 22 \times 0{\cdot}04)\,0{\cdot}04 = 0{\cdot}0448$

$x_2 = (2 - 22 \times 0{\cdot}0448)\,0{\cdot}0448 = 0{\cdot}04544512$

$x_3 = (2 - 22 \times 0{\cdot}04544512)\,0{\cdot}04544512 = 0{\cdot}045454543$

$x_4 = (2 - 22 \times 0{\cdot}04545454)\,0{\cdot}04545454 = 0{\cdot}045454545$

$\therefore$ The required value of $\frac{1}{22}$ is 0·04545454, Correct to seven significant figures.

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Problem 5. Solve the following equations by NR method

(i) $x \sin x + \cos x = 0$ Ans 2·7984

(ii) $2x - 3\sin x - 5 = 0$ Ans 2·88323
