Simple Eigen value Problems

Let us consider the equation

$$\frac{d^2y}{dx^2} + \lambda y = 0 \quad \dashrightarrow (1)$$

which satisfies the boundary conditions

$$y(0) = 0 \quad \text{and} \quad y(\pi) = 0 \quad \dashrightarrow (2)$$

Here λ is a parameter which can assume any real value. we are interested to find the values of λ for which the equation (1) can be solved, satisfying the conditions in (2).

Initial value problems (I.V.P):

In ~~these~~ I.V. problems we sought the solution of a second order differential equation that satisfies two conditions at a single value of the independent variable.

Boundary value problems (B.V.P):

In B.V.P. we sought the solution of a second order differential equation that satisfies two conditions at two distinct values of the independent variable.

Note: usually the number supplementary conditions is equal to the order of the differential equation.

Example 1)- Find the eigen-values λ_n and the eigen-functions $y_n(x)$ for the differential equation

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

Satisfying the boundary conditions

$$y(0) = 0 \text{ and } y(\pi) = 0$$

Solution: Auxiliary equation is $m^2 + \lambda = 0$

$\therefore m^2 = -\lambda$

Case-I when $\lambda = 0$, the solution is

$$y = A + Bx.$$

$y(0) = 0 \Rightarrow A + B\times 0 = 0 \Rightarrow A = 0$

$y(\pi) = 0 \Rightarrow A \cancel{+} B\pi = 0 \Rightarrow B\pi = 0 \quad (\because A = 0)$

$\therefore B = 0$

$\therefore$ The solution is $y = 0$ which is trivial.

Case-II when $\lambda < 0$, (Here $m = \pm\sqrt{-\lambda}$)

$\therefore$ The Solution is $y = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$

$y(0)=0 \Rightarrow A+B=0 \Rightarrow B=-A$

$y(\pi)=0 \Rightarrow Ae^{\sqrt{-\lambda}\pi} + Be^{-\sqrt{-\lambda}\pi} = 0$

$\Leftrightarrow A(e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi}) = 0$ $[\because e^{\sqrt{-\lambda}\pi} - e^{-\sqrt{-\lambda}\pi} \neq 0]$

$\therefore A=0$, Consequently, $B=0$

$\therefore$ The Solution becomes $y=0$ which is again trivial.

Case-III when $\lambda > 0$, (Here $m = \pm i\sqrt{\lambda}$)

$\therefore$ The solution is $y = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$y(0)=0 \Rightarrow A + B\times 0 = 0 \Rightarrow A = 0$

$y(\pi)=0 \Rightarrow A\cos\sqrt{\lambda}\pi + B\sin\sqrt{\lambda}\pi = 0 \Rightarrow B\sin\sqrt{\lambda}\pi = 0$

For a non-trivial Solution, $B \neq 0$

$\therefore \sin\sqrt{\lambda}\pi = 0 = \sin n\pi$, $n=1,2,3,\ldots$

$\therefore \lambda = n^2$, $n=1,2,3$

$\therefore$ The eigen values are $\lambda_n = n^2$, $n=1,2,3,\ldots$

ie $1^2, 2^2, 3^2, \ldots$

and the Corresponding eigen functions are

$y_n(x) = B_n\sin(nx)$, $n=1,2,3,\ldots$

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observation:

1) The eigen values form an increasing sequence of positive numbers which approaches to ∞.

Example-2) Find the eigen values and eigen-functions for the differential equation

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

Satisfying the boundary Conditions (B.C.) $y(0)=0$ and $y(1)=0$

Example-3) Find the eigen values and eigen functions of

$$\frac{d^2y}{dx^2} + \lambda y = 0$$ where $y(0)=0$ and $y'(\pi)=0$.

Example 4: Find the eigen values λ_n and eigen functions $y_n(x)$ for the differential equation

$$\frac{d^2y}{dx^2} + \lambda y = 0 \quad (\lambda > 0)$$

(i) Satisfying the boundary Conditions $y(0) = 0 = y(2\pi)$.

(ii) Satisfying the boundary Conditions $y'(0) = 0$, $y'(l) = 0$.

Example 5: Find the eigen values λ_n and eigen functions $y_n(x)$ for the differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0, \quad (\lambda > 0)$$

(i) Satisfying the B.C. $y(1) = 0 = y(e^{\pi})$

(ii) Satisfying the B.C. $y'(1) = 0 = y'(e^{\pi/2})$