Test for the Equality of means of several normal populations :

Let x_{ij}, $(j=1,2,\cdots n_i, i=1,2,\cdots k)$ be k independent random samples from k normal populations with means $\mu_1, \mu_2, \cdots, \mu_k$ respectively and unknown but common variance σ^2. We want to test the null hypothesis

$$H_0: \mu_1 = \mu_2 = \cdots\cdots = \mu_k = \mu \text{ (say), (unspecified)}.$$
$$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2 \text{ (say), (")}$$

against the alternative hypothesis

$$H_1: \mu_i\text{'s are not equal},$$
$$\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2 \text{ (unspecified)}.$$

We have the parametric space

$$\Omega = \{(\mu_1, \mu_2, \cdots, \mu_k, \sigma^2), -\infty < \mu_i < \infty \ (i=1,2,\cdots k), 0 < \sigma^2 < \infty\}$$

and the sub space under H_0.

$$\omega = \{(\mu_1, \mu_2 \cdots \mu_k, \sigma^2), -\infty < \mu_i = \mu < \infty . (i=1,2,\cdots k), 0 < \sigma^2 < \infty\}.$$

~~where $\sum n_i = n$ (say)~~ :

The likelihood function of the sample observations is

$$L = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_i \sum_j (x_{ij} - \mu_i)^2}$$

where $\sum n_i = n$

For variations of μ_i $(i=1,2,\cdots k)$ and σ^2 in Ω, the m.l.e of μ_i's and σ^2 are

$$\hat{\mu}_i = \bar{x}_i \quad (i=1,2,\cdots k)$$

$$\text{and } \hat{\sigma}^2 = \frac{\sum_i \sum_j (x_{ij} - \bar{x}_i)^2}{n} = \frac{S_W}{n} \text{ (say)}.$$

Where S_W is known as the within sample sum of squares.

$$\text{Hence } L(\hat{\Omega}) = \left(\frac{n}{2\pi S_W}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}$$

In ω. the only variable parameters are μ and σ^2 and we have

$$L(\omega) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \cdot e^{-\frac{1}{2\sigma^2} \sum\sum (x_{ij} - \mu)^2}$$

The mle's of μ and σ^2 are given by

$$\hat{\mu} = \frac{1}{n} \sum_i \sum_j x_{ij} = \frac{1}{n} \sum_i n_i \bar{x}_i = \bar{x}$$

$$\text{and } \hat{\sigma}^2 = \frac{\sum\sum (x_{ij} - \bar{x})^2}{n} = \frac{S_T}{n}, \text{(say)}$$

Where S_T is known as total sum of squares.

$$\text{Hence } L(\hat{\omega}) = \left(\frac{n}{2\pi S_T}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}$$

Now the likelihood ratio criterion λ is

$$\lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} = \left(\frac{S_W}{S_T}\right)^{\frac{n}{2}}$$

Now we have.
$$S_T = \sum_i \sum_j (x_{ij} - \bar{x})^2$$
$$= \sum_i \sum_j \left\{(x_{ij} - \bar{x}_i) + (\bar{x}_i - \bar{x})\right\}^2$$
$$= \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 + \sum_i \sum_j (\bar{x}_i - \bar{x})^2 + 2 \sum_i \sum_j \{(x_{ij} - \bar{x}_i) (\bar{x}_i - \bar{x})$$
$$= \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 + \sum_i n_i (\bar{x}_i - \bar{x})^2 + 2 \sum_i (\bar{x}_i - \bar{x}) \sum_j (x_{ij} - \bar{x}_i)$$

But $\sum_j (x_{ij} - \bar{x}_i) = 0$, being the sum of the ~~ob~~ deviations of the observations of the i^{th} sample from its mean

$$\therefore S_T = \sum_i \sum_j (x_{ij} - \bar{x}_i)^2 + \sum_i n_i (\bar{x}_i - \bar{x})^2$$
$$= S_W + S_B$$

Where $S_B = \sum_i n_i (x_i - \bar{x})^2$ is known as between sample sum of squares.

Hence we have

$$\lambda = \left(\frac{S_W}{S_W + S_B}\right)^{\frac{n}{2}}$$

$$= \frac{1}{\left(1 + \frac{S_B}{S_W}\right)^{\frac{n}{2}}}$$

We know that under H_0 the statistic

$$F = \frac{S_B/(k-1)}{S_W/(n-k)}$$

follows F-distribution with $(k-1, n-k)$ d.f.

Hence $\lambda = \left[1 + \frac{k-1}{n-k} F\right]^{-\frac{n}{2}}$

The critical region of the LR test viz. $0 < \lambda < \lambda_0$ is equivalent to

$$\left[1 + \frac{k-1}{n-k} F\right]^{-\frac{n}{2}} \leq \lambda_0$$

$$\Rightarrow 1 + \frac{k-1}{n-k} F > \lambda_0^{-\frac{2}{n}}.$$

$$\Rightarrow F > \frac{n-k}{k-1}\left[\lambda_0^{-\frac{2}{n}} - 1\right] = A \text{ (say)}.$$

Where A is determined from the equation.

$$P[F > A / H_0] = \alpha$$

Since F follows F-distribution with $(k-1, n-k)$ d.f. we get.

$$A = F_{k-1, n-k}(\alpha)$$

Where $F_{k-1, n-k}(\alpha)$ denotes the upper α point of the F-distribution with $(k-1, n-k)$ d.f.

Hence to test $H_0: \mu_1 = \mu_2 = \cdots = \mu_k = \mu$, $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2$, against H_1: μ_i's are not equal, $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2 > 0$ we reject H_0 if $F > F_{k-1, n-k}(\alpha)$, otherwise H_0 may be accepted.