UNIT 8 TWO-WAY ANOVA WITH m OBSERVATIONS PER CELL

Structure

8.1 INTRODUCTION

In the analysis of variance technique, if explanatory variable is only one and different levels of independent variable is under consideration then it is called one-way analysis of variance and a test of hypothesis is developed for the equality of several mean of different levels of a factor/independent variable/ explanatory variable. But if we are interested to consider two independent variables for analysis in place of one, and able to perform the two hypotheses for the levels of these factors independently (there is no interaction between these two factors). The above analysis has been given in the Units 6 and 7 respectively. But if we are interested to test the interaction between two factors and we have repeated observations then the two-way analysis of variance with m observation per cell is considered. If there are exactly same numbers of observations in the cell then it is called balance.

In this unit, a mathematical model for two-way classified data with m-observations per cell is given in Section 8.2. The basic assumptions are given in Section 8.3 whereas the estimation of parameters is given in Section 8.4. Test of hypothesis for two-way ANOVA is explained in Section 8.5 and degrees of freedom of various sum of squares are described in Section 8.6. The expected values of sum of squares for two factors and their interactions are derived in Section 8.7.

Objectives

After studying this unit, you would be able to

- describe the ANOVA model for two-way classified data with m observations per cell;
- describe the basic assumptions for the given model;
- obtain the estimates of the parameters of the given model;

- describe the test of hypothesis for two-way classified data with m observations per cell;
- derive the expectations of the various sum of squares; and
- perform to test the hypothesis for two-way classified data with m observations per cell.

8.2 ANOVA MODEL FOR TWO-WAY CLASSIFIED DATA WITH m OBSERVATIONS PER CELL

In Unit 7, it was seen that we cannot obtain an estimate of, or make a test for the interaction effect in the case of two-way classified data with one observation per cell. This is possible, however, if some or all of the cells contain more than one observations. We shall assume that there is an equal number of (m) observations in each cell. The m observations in the (i, j)th cell will be denoted y_{ij1}, y_{ij2}, . . ., y_{ijm}. Thus, y_{ijk} is the kth observation for ith level of factor A and jth level of factor B, i = 1, 2, …, p; j = 1, 2, …,q & k = 1, 2, …, m.

The mathematical model

$$y_{ijk} = \mu_{ij} + e_{ijk}$$

where μ_{ij} is the true value for the (i, j)th cell and e_{ijk} is the error. e_{ijk} are assumed to be independently identical normally distributed, each with mean zero and variance σ_e^2. The table of observations can be displayed as follows:

A/B	B_1	B_2	B_j	…	B_q	Total	Total
A_1	y_{111}	y_{121}	y_{1j1}	…	y_{1q1}	$y_{1.1}$	$y_{1..}$
	y_{112}	y_{122}	y_{1j2}	…	y_{1q2}	$y_{1.2}$	
	.	.	.		.	.	
	.	.	.		.	.	
	.	.	.		.	.	
	y_{11m}	y_{12m}	y_{1jm}	…	y_{1qm}	$y_{1.m}$	
A_2	y_{211}	y_{221}	y_{2j1}	…	y_{2q1}	$y_{2.1}$	$y_{2..}$
	y_{212}	y_{222}	y_{2j2}	…	y_{2q2}	$y_{2.2}$	
	.	.	.		.	.	
	.	.	.		.	.	
	.	.	.		.	.	
	y_{21m}	y_{22m}	y_{2jm}	…	Y_{2qm}	$y_{2.m}$	
.	y_{i11}	y_{i21}	y_{ij1}	…	y_{iq1}	$y_{i.1}$	$y_{i..}$
.	y_{i12}	y_{i22}	y_{ij2}	…	y_{iq2}	$y_{i.2}$	
.	.	.	.	…	.	.	
A_i	.	.	.	…	.	.	
.	.	.	.	…	.	.	
.	y_{i1m}	y_{i2m}	y_{ijm}	…	y_{iqm}	$y_{i.m}$	
A_p	y_{p11}	y_{p21}	y_{pj1}	…	y_{pq1}	$y_{p.1}$	$y_{p..}$
	y_{p12}	y_{p22}	y_{pj2}	…	y_{pq2}	$y_{p.2}$	
	.	.	.		.	.	
	.	.	.		.	.	
	.	.	.		.	.	
	y_{p1m}	y_{p2m}	y_{pjm}	…	y_{pqm}	$y_{p.m}$	
Total	$y_{.1.}$	$y_{.2.}$	$y_{.j.}$	…	$y_{.q.}$	$y_{...}$	

The model can be written as

$$y_{ijk} = \mu + (\mu_{i.} - \mu) + (\mu_{.j} - \mu) + (\mu_{ij} - \mu_{i.} - \mu_{.j} + \mu) + e_{ijk}$$

$$= \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk}$$

where, μ is general mean effect, α_i is the effect of i^{th} level of the factor A, β_j is the effect of j^{th} level of factor B, $(\alpha\beta)_{ij}$ is the interaction effect between i^{th} level of A factor and j^{th} level of B factor.

$$\sum_{i=1}^{p}\alpha_i = 0, \sum_{j=1}^{q}\beta_j = 0, \sum_{i=1}^{p}(\alpha\beta)_{ij} = 0, \sum_{j=1}^{q}(\alpha\beta)_{ij} = 0$$

where,

$y_{...}$ = Sum of all the observations.

$y_{i..}$ = Total of all observations in the i^{th} level of factor A

$y_{.j.}$ = Total of all observations in the j^{th} level of factor B.

8.3 BASIC ASSUMPTIONS

Following assumptions should be followed for valid and reliable test procedure for testing of hypothesis as well as for estimation of parameters

1. All the observations y_{ijk} are independent.
2. Different effects are additive in nature.
3. e_{ijk} are independent and identicaly distributed as normal with mean zero and constant variance σ_e^2.

8.4 ESTIMATION OF PARAMETERS

The least square estimates for various effects, obtained by minimizing the residual sum of squares

$$E = \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m}\left[y_{ijk} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij}\right]^2$$

by partially differentiating E with respect to μ, α_i (i=1, 2, ..., p), β_j (i=1, 2, ..., q) and $(\alpha\beta)_{ij}$ for all i = 1, 2, ..., p; j =1, 2, … , q and equating these equations equal to zero. These equations are called normal equations. Solution of these normal equations provide the estimates of these parameters [μ, α_i, β_j, $(\alpha\beta)_{ij}$].

$$\frac{\partial E}{\partial \mu} = -2\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m}\left[y_{ijk} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij}\right] = 0$$

$$\frac{\partial E}{\partial \alpha_i} = -2\sum_{j=1}^{q}\sum_{k=1}^{m}\left[y_{ijk} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij}\right] = 0$$

$$\frac{\partial E}{\partial \beta_j} = -2\sum_{i=1}^{p}\sum_{k=1}^{m}\left[y_{ijk} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij}\right] = 0$$

$$\frac{\partial E}{\partial (\alpha\beta)_{ij}} = -2\sum_{k=1}^{m}\left[y_{ijk} - \mu - \alpha_i - \beta_j - (\alpha\beta)_{ij}\right] = 0$$

These equations give, $\hat{\mu} = \dfrac{\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m} y_{ijk}}{pqm} = \bar{y}_{...}$

$$\hat{\alpha}_i = \frac{\sum_{j=1}^{q}\sum_{k=1}^{m} y_{.jk}}{qm} - \hat{\mu} = \bar{y}_{i..} - \bar{y}_{...}$$

Similarly, $\hat{\beta}_j = \dfrac{\sum_{i=1}^{p}\sum_{k=1}^{m} y_{i.k}}{pm} - \hat{\mu} = \bar{y}_{.j.} - \bar{y}_{...}$

$$\hat{(\alpha\beta)}_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}$$

Substituting the values of $\hat{\mu}_i$, $\hat{\alpha}_i$, $\hat{\beta}_j$ and $(\alpha\hat{\beta})_{ij}$, in the model and then select the value of e_{ijk} such that both the sides are equal, so

$$y_{ijk} = \bar{y}_{...} + (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) + (y_{ijk} - \bar{y}_{ij.})$$

or

$$y_{ijk} - \bar{y}_{...} = (\bar{y}_{i..} - \bar{y}_{...}) + (\bar{y}_{.j.} - \bar{y}_{...}) + (\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}) + (y_{ijk} - \bar{y}_{ij.})$$

Squaring and summing both the sides over i, j & k, then we get

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m}(y_{ijk} - \bar{y}_{...})^2 = mq\sum_{i=1}^{p}(\bar{y}_{i..} - \bar{y}_{...})^2 + mp\sum_{j=1}^{q}(\bar{y}_{.j.} - \bar{y}_{...})^2$$

$$+ m\sum_{i=1}^{p}\sum_{j=1}^{q}(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2 + \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m}(y_{ijk} - \bar{y}_{ij.})^2$$

as usual product terms vanish.

Total Sum of Squares = Sum of Squares due to Factor A+ Sum of Squares due to Factor B + Sum of Squares due to Interaction A and B + Sum of Squares due to Error

or TSS = SSA + SSB + SSAB + SSE

8.5 TEST OF HYPOTHESIS

There are three hypotheses which are to be tested are as follows:

H_{0A}: $\alpha_1 = \alpha_2 = \ldots = \alpha_p = 0$

H_{1A}: $\alpha_1 \neq \alpha_2 \neq \ldots \neq \alpha_p \neq 0$

H_{0B}: $\beta_1 = \beta_2 = \ldots = \beta_q = 0$

H_{1B}: $\beta_1 \neq \beta_2 \neq \ldots \neq \beta_q \neq 0$

H_{0AB}: $(\alpha\beta)_{ij} = 0$ for all i and j or A and B are independent to each other

H_{1AB}: $(\alpha\beta)_{ij} \neq 0$

The appropriate test statistics for testing the above hypothesis is:

$$F = \frac{SSA/(p-1)}{SSE/pq(m-1)} = \frac{MSSA}{MSSE}$$

If this value of F is greater than the tabulated value of F with [(p - 1), pq (m - 1)] df at α level of significance so we reject the null hypothesis, otherwise we may accept the null hypothesis.

Similarly, test statistics for second and third hypotheses are

$$F = \frac{SSB/(q-1)}{SSE/pq(m-1)} = \frac{MSSB}{MSSE}$$

$$F = \frac{SS\,AB/(p-1)(q-1)}{SSE/pq(m-1)} = \frac{MSSAB}{MSSE}$$

For practical point of view, first we should decide whether or not H_{0AB} can be rejected at an appropriate level of significance by using above F. If interaction effects are not significant i.e. the factor A and factor B are independent then we can find the best level of A and best level of B by multiple comparison method using t-test. On the other hand, if they are found to be significant, there may not be a single level of factor A and single level of factor B that will be the best in all situations. In this case, one will have to compare for each level of B at the different levels of A and for each level of A at the different levels of B.

The above analysis can be shown in the following ANOVA table:

ANOVA Table for Two-way Classified Data with m Observations per Cell

Sourses of Variation	**DF**	**SS**	**MSS**	**F**
Between the levels of A	p - 1	$SSA = mq\sum_{i=1}^{p}(\bar{y}_{i..} - \bar{y}_{...})^2$	MSSA = SSA / (p - 1)	F = MSSA / MSSE
Between the levels of B	q - 1	$SSB = mp\sum_{j=1}^{q}(\bar{y}_{.j.} - \bar{y}_{...})^2$	MSSB = SSB / (q - 1)	F = MSSB / MSSE
Interaction AB	(p - 1) (q - 1)	SSAB = $m\sum_{i=1}^{p}\sum_{j=1}^{q}(y_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...})^2$	MSSAB = SSAB / (p - 1)(q - 1)	F = MSS(AB) / MSSE
Error	pq (m - 1)	$TSS = \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m}(y_{ijk} - \bar{y}_{ij.})^2$	MSSE = SSE / pq (m - 1)	
Total	mpq - 1			

Steps for Calculating Various Sums of Squares

1. Calculate G = Grand Total = Total of all observations = $\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m} y_{ijk}$
2. Determine N = Numebr of observations.
3. Find Correction Factor (CF) = G^2/N
4. Raw Sum of Squares (RSS) = $\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m} y_{ijk}^2$
5. Total Sum of Squares (TSS) = RSS - CF
6. Sum of Squares due to Factor A (SSA)
   $= \{y_{1..}^2/mq + y_{2..}^2/mq + ... + y_{i..}^2/mq + ... + y_{p..}^2/mq\} - CF$
7. Sum of Squares due to Factor B (SSB)
   $= \{y_{.1.}^2/mp + y_{.2.}^2/mp + \dots + y_{.j.}^2/mp + ... + y_{.q.}^2/mp\} - CF$
8. Sum of Squares due to Means (SSM)
   $= \{y_{..1}^2/pq + y_{..2}^2/pq + \dots + y_{..k}^2/pq + ... + y_{..m}^2/pq\} - CF$
9. Sum of Squares due to Interation AB(SSAB) = SSM – SSA - SSB
10. Sum of Squares due to Error (SSE) = TSS -SSA-SSB-SSAB
11. Calculate MSSA = SSA/df
12. Calculate MSSB = SSB/df
13. Calculate MSSAB = SS(AB)/df
14. Calculate MSSE = SSE/df
15. Calculate F_A = MSSA/MSSE ~ $F_{(p-1),\ pq(m-1)}$
16. Calculate F_B = MSSB/MSSE ~ $F_{(q-1),\ pq(m-1)}$
17. Calculate F_{AB} = MSS(AB)/MSSE ~ $F_{((p-1)(q-1),\ pq(m-1)}$

8.6 DEGREES OF FREEDOM OF VARIOUS SUM OF SQUARES

Total sum of squares (TSS) considers the pqm observations so the degrees of freedom for TSS are (pqm - 1). One degree of freedom is lost due to the restriction that $\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m}(y_{ijk} - \bar{y}_{...}) = 0$.

The degrees of freedom for sum of squares due to factor A is (p - 1) because it has p levels. Similarly, the degrees of freedom for sum of squares due to factor B is (q - 1) because it has q levels, under consideration. Sum of squares due to interaction of factors A and B is (p - 1) (q - 1) and the degrees of freedom for sum of squares due to errors is pq (m - 1). Thus partitioning of degrees of freedom is as follows:

$$(mpq - 1) = (p - 1) + (q - 1) + (p - 1)(q - 1) + pq\,(m - 1)$$

which implies that the df are additive.

8.7 EXPECTATIONS OF VARIOUS SUM OF SQUARES

8.7.1 Expected Value of Sum of Squares due to Factor A

$$E\,(SSA) = E\left[mq\sum_{i=1}^{p}\left(\bar{y}_{i..} - \bar{y}_{...}\right)^2\right]$$

Substituting the value of $\bar{y}_{i..}$ and $\bar{y}_{...}$ from the model, we get

$$= E\left[mq\sum_{i=1}^{p}\left(\alpha_i + \bar{e}_{i..} - \bar{e}_{...}\right)^2\right]$$

or
$$E\,(SSA) = E\left[mq\sum_{i=1}^{p}\left\{\alpha_i^2 + \left(\bar{e}_{i..} - \bar{e}_{...}\right)^2 + 2\alpha_i\left(\bar{e}_{i..} - \bar{e}_{...}\right)\right\}\right]$$

$$= mq\sum_{i=1}^{p} E\left\{\alpha_i^2 + \left(\bar{e}_{i..} - \bar{e}_{...}\right)^2 + 2\alpha_i\left(\bar{e}_{i..} - \bar{e}_{...}\right)\right\}$$

$$= mq\sum_{i=1}^{p}\alpha_i^2 + mqE\left\{\sum_{i=1}^{p}\left(\bar{e}_{i..} - \bar{e}_{..}\right)^2\right\} + 0$$

$$\left[\text{Because } E\left(\bar{e}_{i..} - \bar{e}_{..}\right) = 0\right]$$

$$\therefore E(SSA) = mq\sum_{i=1}^{p}\alpha_i^2 + mq\,E\left[\sum_{i=1}^{p}\left(\bar{e}_{i..}\right)^2 - p\,\bar{e}_{...}^2\right]$$

$$= mq\sum_{i=1}^{p}\alpha_i^2 + mq\left[\sum_{i=1}^{p}E\left(\bar{e}_{i..}\right)^2 - p\,E\left(\bar{e}_{...}^2\right)\right]$$

$$= mq\sum_{i=1}^{p}\alpha_i^2 + mq\left[\sum_{i=1}^{p}\frac{\sigma_e^2}{mq} - p\,\frac{\sigma_e^2}{mpq}\right]$$

$$= mq\sum_{i=1}^{p}\alpha_i^2 + p\,\sigma_e^2 - \sigma_e^2$$

$$\therefore\ E\left(SSA\right) = mq\sum_{i=1}^{p}\alpha_i^2 + (p-1)\,\sigma_e^2$$

$$\Rightarrow E\left(\frac{SSA}{p-1}\right) = \frac{mq}{(p-1)}\sum_{i=1}^{p}\alpha_i^2 + \sigma_e^2$$

or
$$E\left(MSSA\right) = \sigma_e^2 + \frac{mq}{(p-1)}\sum_{i=1}^{p}\alpha_i^2$$

Under H_{0A} the MSSA is an unbiased estimate of σ_e^2.

8.7.2 Expected Value of Sum of Squares due to Factor B

Proceeding similarly, or by symmetry, we have

$$E(SSB) = E\left[mp\sum_{j=1}^{q}\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2\right]$$

Substituting the value of $\bar{y}_{.j.}$ and $\bar{y}_{...}$ from the model

we get $$E(SSB) = E\left[mp\sum_{i=1}^{q}\left(\beta_j + \bar{e}_{.j.} - \bar{e}_{...}\right)^2 \right]$$

or $$E(SSB) = \left[mp\sum_{i=1}^{q}\left\{\beta_j^2 + \left(\bar{e}_{.j.} - \bar{e}_{...}\right)^2 + 2\beta_j\left(\bar{e}_{.j.} - \bar{e}_{...}\right)\right\}\right]$$

or $$E(SSB) = mp\sum_{j=1}^{q}\beta_j^2 + mp\,E\left[\sum_{j=1}^{q}\left(\bar{e}_{.j.} - \bar{e}_{...}\right)^2\right] + 0$$

or $$E(SSB) = mp\sum_{j=1}^{q}\beta_j^2 + mp\,E\left[\sum_{j=1}^{q}\left(\bar{e}_{.j.}\right)^2 - q\,\bar{e}_{...}^2\right]$$

$$= mp\sum_{j=1}^{q}\beta_j^2 + mp\left[\sum_{j=1}^{q}E\left(\bar{e}_{.j.}\right)^2 - qE\left(\bar{e}_{...}^2\right)\right]$$

$$= mp\sum_{j=1}^{q}\beta_j^2 + mp\left[\sum_{j=1}^{q}\frac{\sigma_e^2}{mp} - q\frac{\sigma_e^2}{mpq}\right]$$

$$= mp\sum_{j=1}^{q}\beta_j^2 + q\,\sigma_e^2 - \sigma_e^2$$

$$E(SSB) = \left(q-1\right)\sigma_e^2 + mp\sum_{j=1}^{q}\beta_j^2$$

$$E\left(\frac{SSB}{q-1}\right) = \sigma_e^2 + \frac{mp}{\left(q-1\right)}\sum_{j=1}^{q}\beta_j^2$$

or $$E(MSSB) = \sigma_e^2 + \frac{mp}{\left(q-1\right)}\sum_{j=1}^{q}\beta_j^2$$

Under H_{0B} the MSSB is an unbiased estimate of σ_e^2. Similarly you can obtain the expected value of SSAB, which will be

$$E\ \left(SSAB\right) = m\sum_{i=1}^{p}\sum_{j=1}^{q}\left(\alpha\beta\right)_{ij}^2 + \left(p-1\right)\left(q-1\right)\sigma_e^2$$

$$E\left[\frac{SSAB}{\left(p-1\right)\left(q-1\right)}\right] = \sigma_e^2 + \frac{m}{\left(p-1\right)\left(q-1\right)}\sum_{i=1}^{p}\sum_{j=1}^{q}\left(\alpha\beta\right)_{ij}^2$$

or $$E\ (MSSAB) = \sigma_e^2 + \frac{m}{\left(p-1\right)\left(q-1\right)}\sum_{i=1}^{p}\sum_{j=1}^{q}\left(\alpha\beta\right)_{ij}^2$$

Under H_{0AB}, the mean sum of squares due to interaction between Factor A and B is an unbiased estimate of σ_e^2.

8.7.3 Expected Value of Sum of Squares due to Error

Proceeding similarly, or by symmetry, we have

$$E\,(SSE) = E\left(\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m}\left(y_{ijk} - \bar{y}_{ij.}\right)^2\right)$$

Substituting the value of y_{ijk} and $\bar{y}_{ij.}$ from the model, we have

$$E(SSE) = E\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m}\left(e_{ijk} - \bar{e}_{ij.}\right)^2\right]$$

$$= E\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left(\sum_{k=1}^{m}\left(e_{ijk} - \bar{e}_{ij.}\right)^2\right)\right]$$

$$= E\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\left(\sum_{k=1}^{m} e_{ijk}^2 - m\,\bar{e}_{ij.}^2\right)\right]$$

$$= E\left[\sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m} e_{ijk}^2 - m\sum_{i=1}^{p}\sum_{j=1}^{q}\bar{e}_{ij.}^2\right]$$

$$= \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{m} E\left(e_{ijk}^2\right) - m\sum_{i=1}^{p}\sum_{j=1}^{q} E\left(\bar{e}_{ij.}^2\right)$$

$$= mpq\ \sigma_e^2 - mpq\,\sigma_e^2 / m$$

$$= (mpq - 1)\ \sigma_e^2$$

or $\quad E\left(\dfrac{SSE}{pq\,(m-1)}\right) = \sigma_e^2$

or $\quad E\,(MSSE) = \sigma_e^2$

Hence, mean sum of squares due to error is an unbiased estimate of σ_e^2.

Example 1: A manufacturer wishes to determine the effectiveness of four types of machines (A, B, C and D) in the production of bolts. To accumulate this, the numbers of defective bolts produced for each of two shifts in the results are shown in the following table:

Machine	First shift					Second Shift				
	M	T	W	Th	F	M	T	W	Th	F
A	6	4	5	5	4	5	7	4	6	8
B	10	8	7	7	9	7	9	12	8	8
C	7	5	6	5	9	9	7	5	4	6
D	8	4	6	5	5	5	7	9	7	10

Perform an analysis of variance to determine at 5% level of significance, whether there is a difference (a) Between the machines and (b) Between the shifts.

Solution: There are two factors the machine and shift. The levels of machine are four and levels of shift are two. The Computation results are as follows:

G = 6+ 4 + 5 + 5 + 4 + 5 + 7 + 4 + 6 + 8 + 10 + 8 + 7 + 7 + 9 + 7 + 9 + 12 + 8 + 8 + 7 + 5 + 6 + 5 + 9 + 9 + 7 + 5 + 4 + 6 + 8 + 4 + 6 + 5 + 5 + 7 + 9 + 7 + 10

= 268

N = 40

$$CF = \frac{G^2}{N} = \frac{268 \times 268}{40} = 1795.6$$

Raw Sum of Squares (RSS) = $6^2 + 4^2 + \ldots\ldots + 10^2$ = 1946

Total Sum of Squares (TSS) = RSS - CF = 1946 - 1795.6 = 150.4

Sum of square due to machines and due to shifts can be calculated by considering the following two-way table:

Machine	Shift		Total
	I Shift	II Shift	
A	24	30	54
B	41	44	85
C	32	31	63
D	28	38	66
Total	125	143	268

Sum of Squares due to Machine $(SSM) = \frac{54^2}{10} + \frac{85^2}{10} + \frac{63^2}{10} + \frac{66^2}{10} - CF$

= 1846.6 − 1795.6 = 51.0

Sum of Squares due to Shifts $(SSS) = \frac{125^2}{20} + \frac{143^2}{20} - CF$

= 1803.7 - 1795.6 = 8.1

Sum of Squares due to Interaction (SSMS)

$$= \frac{(24)^2}{5} + \frac{(41)^2}{5} + \frac{(32)^2}{5} + \frac{(28)^2}{5} + \frac{(30)^2}{5} + \frac{(44)^2}{5} + \frac{(31)^2}{5} + \frac{(38)^2}{5} - CF - (SSM) - (SSS)$$

= 1861.2 − 1795.6 − 51.0 − 8.1 = 6.5

Finaly, the Sum of Squares due to error is founded by subtracting the SSM, SSS and SSSM from TSS

SSE = TSS - SSM − SSS − SSMS

= 150.4 - 51.0 - 8.1 - 6.5 = 84.8

$$\text{MSSM} = \frac{\text{SSM}}{\text{df}} = \frac{51.0}{3} = 17$$

$$\text{MSSS} = \frac{\text{SSS}}{\text{df}} = \frac{8.1}{1} = 8.1$$

$$\text{MSSE} = \frac{\text{SSE}}{\text{df}} = \frac{84.8}{31} = 2.65$$

$$\text{MSSMS} = \frac{\text{SSMS}}{\text{df}} = \frac{6.5}{3} = 2.167$$

For testing H_{0A} : Mean effect of Machine A= Machine B =

Machine C = Machine D, is

$$F = \frac{17}{2.65} = 6.42$$

For testing H_{0B} : Mean effect of Shift A = Shift B, is

$$F = \frac{8.1}{2.65} = 3.06$$

Similarly, for testing H_{0AB} : Interaction effect of Machine and Shift, is

$$F = \frac{2.167}{2.65} = 0.817$$

ANOVA Table for Two-way Classified Data m- Observation per Cell

Sources of Variation	**Degrees of Freedom (DF)**	**Sum of Squares (SS)**	**Mean Sum of Squares (MSS)**	**F-test or Variance Ratio**
Due to Machinery	3	51.0	17	$\frac{17}{2.65} = 6.42$
Due to Shift	1	8.1	8.1	$\frac{8.1}{2.65} = 3.06$
Due to Interaction	3	6.5	2.167	$\frac{2.167}{2.65} = 0.817$
Due to Error	32	84.8	2.65	
Total	39	150.4		

The tabulated value of F at 3 and 32 degrees of freedom at 5% level of significance is 2.90. The computed value of F for interaction is 0.817 so the average performances in different shifts are not significant. There is a significant difference among machines, since the calculated value of F for machines is 6.42 and the critical value (tabulated value) of F is 2.90. The tabulated value for shifts is 4.15. The calculated value of F for shifts is 3.06. Hence, there is no difference due to shifts.

E1) An experiment is performed to determine the effect of two advertising campaigns on three kinds of cake mixes. Sales of each mix were recorded after the first advertising campaigns and then after the second advertising campaign. This experiment was repeated three times for each advertising campaign and got the following results:

	Campaign I	Campaign II
Mix1	574, 564, 550	1092, 1086, 1065
Mix2	524, 573, 551	1028, 1073, 998
Mix3	576, 540, 592	1066, 1045, 1055

Perform an analysis of variance to determine at 5% level of significance, whether there is a difference (a) Between the cake mixes and (b) Between the campaigns.

8.8 SUMMARY

In this unit, we have discussed:

1. The ANOVA model for two-way classified data with m observations per cell;
2. The basic assumptions for the given model;
3. How to obtain the estimates of the parameters of the given model;
4. How to test the hypothesis for two-way classified data with m observations per cell;
5. How to derive the expectations of the various sum of squares; and
6. Numerical problems to test the hypothesis for two-way classified data with m observations per cell.

8.9 SOLUTIONS /ANSWERS

E1) For set up an ANOVA Table for this problem, the computation results are as follows:

Grand Total G = 14552

N = 18

Correction Factor (CF) = $(14552 \times 14552)/18 = 11764483.55$

RSS = 12882026

TSS = 1117542

SSA = 1107070

SSB = 2957

SSAB = 1126

SSE = 6389

ANOVA Table

Sources of Variation	**DF**	**SS**	**MSS**	**F-Calculated**	**F-Tabulated at 5% level of significance**
Advertising campaign	1	1107070	1107070	1107070/532.42 = 2079.32	F(1,12) = 243.9
Cake Mix	2	2957	1478.5	1478.5/532.42 = 2.8	F(2,12) = 19.41
Interaction	2	1126	563	563/532.42 = 1.06	F(2,12) = 19.41
Error	12	6389	532.42		
Total	17	1117542			

Since computed value of F for cake mix and interaction are 2.8 and 1.06 respectively which are less than corresponding tabulated value so they are not significant. Whereas the calculated value of F for advertising campaign is greater than corresponding tabulated value so there is a significant difference among advertising campaign.

TABLE: The F Table

Value of F Corresponding to 5% (Normal Type) and 1% (Bold Type) of the Area in the Upper Tail

Degrees of Freedom: (Denominator)	1	2	3	4	5	6	7	8	9	10	11	12	14	16	20	24	30	∞
1	161	200	216	225	230	234	237	239	241	242	243	244	245	246	248	249	250	254
	4,052	**4,999**	**5,403**	**5,625**	**5,764**	**5,859**	**5,928**	**5,981**	**6,022**	**6,056**	**6,082**	**6,106**	**6,142**	**6,169**	**6,208**	**6,234**	**6,258**	**6,366**
2	18.51	19.00	19.16	19.25	19.30	19.33	19.36	19.37	19.38	19.39	19.40	19.41	19.42	19.43	19.44	19.45	19.46	19.50
	98.49	**99.00**	**99.17**	**99.25**	**99.30**	**99.33**	**99.34**	**99.36**	**99.38**	**99.40**	**99.41**	**99.42**	**99.43**	**99.44**	**99.45**	**99.46**	**99.47**	**99.50**
3	10.13	9.55	9.28	9.12	9.01	8.94	8.88	8.84	8.81	8.78	8.76	8.74	8.71	8.69	8.66	8.64	8.62	8.53
	34.12	**30.82**	**29.46**	**28.71**	**28.24**	**27.91**	**27.67**	**27.49**	**27.34**	**27.23**	**27.13**	**27.05**	**26.92**	**26.83**	**26.69**	**26.60**	**26.50**	**26.12**
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.93	5.91	5.87	5.84	5.80	5.77	5.74	5.63
	22.20	**18.00**	**16.69**	**15.98**	**15.52**	**15.21**	**14.98**	**14.80**	**14.66**	**14.54**	**14.45**	**14.37**	**14.24**	**14.15**	**14.02**	**13.93**	**13.83**	**13.46**
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.78	4.74	4.70	4.68	4.64	4.60	4.56	4.53	4.50	4.36
	16.26	**13.27**	**12.06**	**11.39**	**10.97**	**10.67**	**10.45**	**10.27**	**10.15**	**10.05**	**9.96**	**9.89**	**9.77**	**9.68**	**9.55**	**9.47**	**9.38**	**9.02**
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.96	3.92	3.87	3.84	3.81	3.67
	13.74	**10.92**	**9.78**	**9.15**	**8.75**	**8.47**	**8.26**	**8.10**	**7.98**	**7.87**	**7.79**	**7.72**	**7.60**	**7.52**	**7.39**	**7.31**	**7.23**	**6.88**
7	5.59	4.47	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.63	3.60	3.57	3.52	3.49	3.44	3.41	3.38	3.23
	12.25	**9.55**	**8.45**	**7.85**	**7.46**	**7.19**	**7.00**	**6.84**	**6.71**	**6.62**	**6.54**	**6.47**	**6.35**	**6.27**	**6.15**	**6.07**	**5.98**	**5.65**
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.34	3.31	3.28	3.23	3.20	3.15	3.12	3.08	2.93
	11.26	**8.65**	**7.59**	**7.01**	**6.63**	**6.37**	**6.19**	**6.03**	**5.91**	**5.82**	**5.74**	**5.67**	**5.56**	**5.48**	**5.36**	**5.28**	**5.20**	**4.86**
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.13	3.10	3.07	3.02	2.98	2.93	2.90	2.86	2.71
	10.56	**8.02**	**6.99**	**6.42**	**6.06**	**5.80**	**5.62**	**5.47**	**5.35**	**5.26**	**5.18**	**5.11**	**5.00**	**4.92**	**4.80**	**4.73**	**4.64**	**4.31**
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.97	2.94	2.91	2.86	2.82	2.77	2.74	2.70	2.54
	10.04	**7.56**	**6.55**	**5.99**	**5.64**	**5.39**	**5.21**	**5.06**	**4.95**	**4.85**	**4.78**	**4.71**	**4.60**	**4.52**	**4.41**	**4.33**	**4.25**	**3.91**
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.86	2.82	2.79	2.74	2.70	2.65	2.61	2.57	2.40
	9.65	**7.20**	**6.22**	**5.67**	**5.32**	**5.07**	**4.88**	**4.74**	**4.63**	**4.54**	**4.46**	**4.40**	**4.29**	**4.21**	**4.10**	**4.02**	**3.94**	**3.60**
12	4.75	3.88	3.49	3.26	3.11	3.00	2.92	2.85	2.80	2.76	2.72	2.69	2.64	2.60	2.54	2.50	2.46	2.30
	9.33	**6.93**	**5.95**	**5.41**	**5.06**	**4.82**	**4.65**	**4.50**	**4.39**	**4.30**	**4.22**	**4.16**	**4.05**	**3.98**	**3.86**	**3.78**	**3.70**	**3.36**
13	4.67	3.80	3.41	3.18	3.02	2.92	2.84	2.77	2.72	2.67	2.63	2.60	2.55	2.51	2.46	2.42	2.38	2.21
	9.07	**6.70**	**5.74**	**5.20**	**4.86**	**4.62**	**4.44**	**4.30**	**4.19**	**4.10**	**4.02**	**3.96**	**3.85**	**3.78**	**3.67**	**3.59**	**3.51**	**3.16**
14	4.60	3.74	3.34	3.11	2.96	2.85	2.77	2.70	2.65	2.60	2.56	2.53	2.48	2.44	2.39	2.35	2.31	2.13
	8.86	**6.51**	**5.56**	**5.03**	**4.69**	**4.46**	**4.28**	**4.14**	**4.03**	**3.94**	**3.86**	**3.80**	**3.70**	**3.62**	**3.51**	**3.43**	**3.34**	**3.00**
15	4.54	3.68	3.29	3.06	2.90	2.79	2.70	2.64	2.59	2.55	2.51	2.48	2.43	2.39	2.33	2.29	2.25	2.07
	8.68	**6.36**	**5.42**	**4.89**	**4.56**	**4.32**	**4.14**	**4.00**	**3.89**	**3.80**	**3.73**	**3.67**	**3.56**	**3.48**	**3.36**	**3.29**	**3.20**	**2.87**
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.45	2.42	2.37	2.33	2.28	2.24	2.20	2.01
	8.53	**6.23**	**5.29**	**4.77**	**4.44**	**4.20**	**4.03**	**3.89**	**3.78**	**3.69**	**3.61**	**3.55**	**3.45**	**3.37**	**3.25**	**3.18**	**3.10**	**2.75**
17	4.45	3.59	3.20	2.96	2.81	2.70	2.62	2.55	2.50	2.45	2.41	2.38	2.33	2.29	2.23	2.19	2.15	1.96
	8.40	**6.11**	**5.18**	**4.67**	**4.34**	**4.10**	**3.93**	**3.79**	**3.68**	**3.95**	**3.52**	**3.45**	**3.35**	**3.27**	**3.16**	**3.08**	**3.00**	**2.65**
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.37	2.34	2.29	2.25	2.19	2.15	2.11	1.92
	8.28	**6.01**	**5.09**	**4.58**	**4.25**	**4.01**	**3.85**	**3.71**	**3.60**	**3.51**	**3.44**	**3.37**	**3.27**	**3.19**	**3.07**	**3.00**	**2.91**	**2.57**
19	4.38	3.52	3.13	2.90	2.74	2.63	2.55	2.48	2.43	2.38	2.34	2.31	2.26	2.21	2.15	2.11	2.07	1.88
	8.18	**5.93**	**5.01**	**4.50**	**4.17**	**3.94**	**3.77**	**3.63**	**3.52**	**3.43**	**3.36**	**3.30**	**3.19**	**3.12**	**3.00**	**2.92**	**2.84**	**2.49**
20	4.35	3.49	3.10	2.87	2.71	2.60	2.52	2.45	2.40	2.35	2.31	2.28	2.23	2.18	2.12	2.08	2.04	1.84
	8.10	**5.85**	**4.94**	**4.43**	**4.10**	**3.87**	**3.71**	**3.56**	**3.45**	**3.37**	**3.30**	**3.23**	**3.13**	**3.05**	**2.94**	**2.86**	**2.77**	**2.42**
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.28	2.25	2.20	2.15	2.09	2.05	2.00	1.81
	8.02	**5.78**	**4.87**	**4.37**	**4.04**	**3.81**	**3.65**	**3.51**	**3.40**	**3.31**	**3.24**	**3.17**	**3.07**	**2.99**	**2.88**	**2.80**	**2.72**	**2.36**

Degrees of Freedom (Numerator) across columns.

TABLE (Continued)

Degrees of Freedom: Denominator	Degrees of Freedom: Numerator 1	2	3	4	5	6	7	8	9	10	11	12	14	16	20	24	30	∞
22	4.30	3.44	3.05	2.82	2.66	2.55	2.47	2.40	2.35	2.30	2.23	2.23	2.18	2.13	2.07	2.03	1.98	1.78
	7.94	**5.72**	**4.82**	**4.31**	**3.99**	**3.76**	**3.59**	**3.45**	**3.35**	**3.26**	**3.18**	**3.12**	**3.02**	**2.94**	**2.83**	**2.75**	**2.67**	**2.31**
23	4.28	3.42	3.03	2.80	2.64	2.53	2.45	2.38	3.32	2.28	2.24	2.20	2.14	2.10	2.04	2.00	1.96	1.76
	7.88	**5.66**	**4.76**	**4.26**	**3.94**	**3.71**	**3.54**	**3.41**	**3.30**	**3.21**	**3.14**	**3.07**	**2.97**	**2.89**	**2.78**	**2.70**	**2.62**	**2.26**
24	4.26	3.40	3.01	2.78	2.62	2.51	2.43	2.36	2.30	2.26	2.22	2.18	2.13	2.09	2.02	1.98	1.94	1.73
	7.82	**5.61**	**4.72**	**4.22**	**3.90**	**3.67**	**3.50**	**3.36**	**3.25**	**3.17**	**3.09**	**3.03**	**2.93**	**2.85**	**2.74**	**2.66**	**2.58**	**2.21**
25	4.24	3.38	2.99	2.76	2.60	2.49	2.41	2.34	2.28	2.24	2.20	2.16	2.11	2.06	2.00	1.96	1.92	1.71
	7.77	**5.57**	**4.68**	**4.18**	**3.86**	**3.63**	**3.46**	**3.32**	**3.21**	**3.13**	**3.05**	**2.99**	**2.89**	**2.81**	**2.70**	**2.62**	**2.54**	**2.17**
26	4.22	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.18	2.15	2.10	2.05	1.99	1.95	1.90	1.69
	7.72	**5.53**	**4.64**	**4.14**	**3.82**	**3.59**	**3.42**	**3.29**	**3.17**	**3.09**	**3.02**	**2.96**	**2.86**	**2.77**	**2.66**	**2.58**	**2.50**	**2.13**
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.30	2.25	2.20	2.16	2.13	2.08	2.03	1.97	1.93	1.88	1.67
	7.68	**5.49**	**4.60**	**4.11**	**3.79**	**3.56**	**3.39**	**3.26**	**3.14**	**3.06**	**2.98**	**2.93**	**2.83**	**2.74**	**2.63**	**2.55**	**2.47**	**2.10**
28	4.20	3.34	2.95	2.71	2.56	2.44	2.36	2.29	2.24	2.19	2.15	2.12	2.06	2.02	1.96	1.91	1.87	1.65
	7.64	**5.45**	**4.57**	**4.07**	**3.76**	**3.53**	**3.36**	**3.23**	**3.11**	**3.03**	**2.95**	**2.90**	**2.80**	**2.71**	**2.60**	**2.52**	**2.44**	**2.06**
29	4.18	3.33	2.93	2.70	2.54	2.43	2.35	2.28	2.22	2.18	2.14	2.10	2.05	2.00	1.94	1.90	1.85	1.64
	7.60	**5.42**	**4.54**	**4.04**	**3.73**	**3.50**	**3.33**	**3.20**	**3.08**	**3.00**	**2.92**	**2.87**	**2.77**	**2.68**	**2.57**	**2.49**	**2.41**	**2.03**
30	4.17	3.32	2.92	2.69	2.53	2.42	2.34	2.27	2.21	2.16	2.12	2.09	2.04	1.99	1.93	1.89	1.84	1.62
	7.56	**5.39**	**4.51**	**4.02**	**3.70**	**3.47**	**3.30**	**3.17**	**3.06**	**2.98**	**2.90**	**2.84**	**2.74**	**2.66**	**2.55**	**2.47**	**2.38**	**2.01**
∞	3.84	2.99	2.60	2.37	2.21	2.09	2.01	1.94	1.88	1.83	1.79	1.75	1.69	1.64	1.57	1.52	1.46	1.00
	6.64	**4.60**	**3.78**	**3.32**	**3.02**	**2.80**	**2.64**	**2.51**	**2.41**	**2.32**	**2.24**	**2.18**	**2.07**	**1.99**	**1.87**	**1.79**	**1.69**	**1.00**